Bayesian Estimation Procedures For Three Parameter Exponentiated Weibull Distribution Under Entropy Loss Function And Type II Censoring

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Abstract

Bayes estimators have been obtained for reliability function R(t)=P(X>t) and P=P(X>Y), under type II censoring for three parameter exponentiated Weibull distribution. Consideration is given to entropy loss function. A new technique of obtaining Bayes estimators of these parametric functions is introduced in which major role is played by the estimators of powers of the parameter. Numerical illustrations for these new results are given.

Keywords: Three parameter exponentiated Weibull distribution; type II censoring; Bayes estimators; entropy loss function.

1. Introduction

Bayesian ideas were introduced for the first time in reliability and life testing by Bhattacharya (1967), who considered the problems of estimating the parameter and reliability function of one parameter exponential distribution under type II censoring. Consideration was given to squared error loss function (SELF). Since then, a lot of work has been done in the literature for estimating the parameters and reliability function of various lifetime distributions under SELF. For a brief review, one may refer to the book by Martz and Waller (1982). Another measure of reliability under stress-strength set-up is the probability P=(X>Y), where the random variables (rv's) X and Y represents, respectively, strength and stress. Enis and Geisser (1971) derived Bayes estimators of P under SELF when X and Y were assumed to follow exponential distributions.

While estimating the mean failure time or reliability function, the use of symmetrical loss function is inappropriate because of the recognition of the fact that overestimation is usually more serious than the underestimation [see Basu and Ebrahimi (1991) and Calabria and Pulcini (1996) for a brief discussion]. For the situations when overestimation and underestimation are not equally serious, Varian (1975) has suggested LINEX (linear in exponential) loss function, which has been further used by Zellner (1986). Basu and Ebrahimi (1991) derived Bayes estimators of the mean failure time, reliability function and 'P' of exponential distribution considering both the SELF and LINEX loss functions.

The LINEX loss function is suitable for the estimation of location parameter but not for the estimation of scale parameter and other parametric functions. Calabria and Pulcini (1994) suggested the general entropy loss function (GELF) for estimating these quantities.

The 'exponentiated Weibull family', introduced by Mudholkar and Srivastava (1993) as an extension of the two parameter Weibull family and is obtained by introducing one additional shape parameter. The main feature of this family is that it allows bathtub shaped as well as unimodal hazard rates in addition to various monotone hazard rates. The cumulative distribution function (cdf), probability density function (pdf) and the reliability function R(t) of three parameter exponentiated Weibull distribution are given, respectively, by

$$F\left(x;\alpha,\lambda,\delta\right)=\left(1-e^{-\lambda x^{\delta}}\right)^{\alpha}\quad ;x,\alpha,\lambda,\delta>0,$$

$$f\left(x;\alpha,\lambda,\delta\right)=\alpha\lambda\delta x^{\delta-1}e^{-\lambda x^{\delta}}\left(1-e^{-\lambda x^{\delta}}\right)^{\alpha-1};x,\alpha,\lambda,\delta>0,$$

and

$$R(t)=1-\left(1-e^{-\lambda t^{\delta}}\right)^{\alpha}\quad ;t,\alpha,\lambda,\delta>0, \tag{1.1}$$

where α and δ both are the shape parameters while λ is scale parameter of the family. A detailed discussion of three parameter exponentiated Weibull distribution can be had from Mudholkar et al. (1995). Singh et al. (2002) discussed the classical and Bayesian methods of parameter estimation for complete sample case. Singh et al. (2004) have discussed the classical and Bayesian methods of parameter estimation under type II censoring. This paper is an attempt in the direction of Bayesian estimation of R(t) and 'P' for three parameter exponentiated Weibull distribution under type II censoring.

The purpose of the present paper is manifold. For the distribution given at (1.1), Bayes estimators are derived for the powers (positive, as well as, negative) of the parameter, reliability function R(t) and 'P', under GELF. Type II censoring is considered. Deviating from the conventional methods of obtaining Bayes estimators of R(t) and 'P', Bayes estimators of the powers of the parameter are utilized to obtain Bayes estimator of the cdf/pdf at a specified point. This estimator is subsequently used to obtain Bayes estimators of R(t) and 'P'. Thus, in all the estimation problems, the major role is played by the estimators of the powers of the parameter [see Lemma 1]. Expressions for the risks, posterior risks and Bayes risks of estimators of the powers of the parameter are provided.

In Section 2, we give the set-up of the estimation problems and introduce the notations and definitions. In Section 3, we obtain Bayes estimators of the powers of the parameter, R(t) and 'P'. Finally in Section 4, the simulation study is performed.

2. Set-Up Of The Estimation Problems, Notations And Definitions

Let the random variable (rv) X follow the three parameter exponentiated Weibull distribution, whose pdf is given at (1.1). Throughout, we assume that α is unknown but λ and δ are known.

Suppose n items are put on a test and the test is terminated after the first r ordered observations are recorded. Let $0 \le X_{(1)} \le .... \le X_{(r)}, 0 < r < n;$ be the lifetimes of the first r ordered observations. Obviously, (n-r) items survived until $X_{(r)}$. Denoting by $\underline{x} = (x_1, x_2, ..., x_r)$, $S_r = -\left[\sum_{i=1}^{r} ln\left\{1-e^{-\lambda x_{(i)}^{\delta}}\right\} + (n-r) ln\left\{1-e^{-\lambda x_{(r)}^{\delta}}\right\}\right]$ *and* $L(\alpha \backslash \underline{x})$, the likelihood of observing α, we have

$$L(\alpha \backslash \underline{x}) = \alpha^r \lambda^r \delta^r \left[\prod_{i=1}^{r} x_{(i)}^{\delta} \left\{x^{\delta}{}_{(r)}\right\}^{n-r}\right] exp\left[-\lambda\left(\sum_{i=1}^{r} x_{(i)}^{\delta} + (n-r) x_{(r)}^{\delta}\right)\right]$$
$$\times exp\left[(\alpha\text{-}1)\left[\sum_{i=1}^{r} ln\left\{1-e^{-\lambda x_{(i)}^{\delta}}\right\} + (n-r) ln\left\{1-e^{-\lambda x_{(r)}^{\delta}}\right\}\right]\right],$$

or,

$$L(\alpha \backslash \underline{x}) \propto \alpha^r exp\left[(\alpha\text{-}1)\sum_{i=1}^{r} ln\left\{1-e^{-\lambda x_{(i)}^{\delta}}\right\} + (n-r) ln\left\{1-e^{-\lambda x_{(r)}^{\delta}}\right\}\right],$$

or,

$$L(\alpha|\underline{x}) \propto \alpha^r \, exp[-\alpha s_r]. \tag{2.1}$$

Looking at (2.1), we consider the natural conjugate prior distribution for α to be gamma with pdf

$$\Pi(\alpha) = \frac{\mu^v}{\Gamma v} \alpha^{v-1} exp(-\mu\alpha) \; ; \quad \alpha,\mu > 0 \text{ and } v \text{ is positive integer.} \tag{2.2}$$

Combining (2.1) and (2.2) via Bayes theorem, the posterior density of α comes out to be

$$h(\alpha|s_r) = \frac{(s_r + \mu)^{r+v}}{\Gamma(r+v)} \alpha^{r+v-1} exp\{-(s_r + \mu)\alpha\}. \tag{2.3}$$

In order to estimate 'P', let n items on X and m items on Y are put through a life test and r and s being their truncation numbers, respectively. The rv X has pdf $f(x;\alpha_1,\lambda_1,\delta_1)$ and the rv Y has pdf $f(y;\alpha_2,\lambda_2,\delta_2)$. Let us denote by $S_r = -\left[\sum_{i=1}^{r} ln\left\{1-e^{-\lambda_1 x_{(i)}^{\delta_1}}\right\} + (n-r) ln\left\{1-e^{-\lambda_1 x_{(r)}^{\delta_1}}\right\}\right]$ and $T_s = -\left[\sum_{j=1}^{s} ln\left\{1-e^{-\lambda_2 y_{(j)}^{\delta_2}}\right\} + (m-s) ln\left\{1-e^{-\lambda_2 y_{(s)}^{\delta_2}}\right\}\right]$.

Here, we assume that $\lambda_1, \lambda_2, \delta_1$ *and* δ_2 are known but α_1 and α_2 are unknown. We consider the conjugate priors for α_1 and α_2 given at (2.2) with parameters (μ_1,v_1) *and* (μ_2,v_2), respectively.

Let us make the transformation $U=-log\left(1-e^{-\lambda X^{\delta}}\right)$. It is easy to see that U follows exponential distribution with pdf

$$f(u;\alpha) = \alpha exp(-\alpha u); u > 0.$$

If we consider the transformation $Z_i = (n-i+1)[U_{(i)} - U_{(i-1)}],\ i = 1,2,...,r$, then Z_i's are independent and identically distributed rv's, each having exponential distribution.

Moreover, since $\sum_{i=1}^{r} Z_i = S_r$, from the additive property of exponential distribution, the pdf of S_r is

$$h(s_r;\alpha) = \frac{\alpha^r}{\Gamma(r)} s_r^{r-1} exp(-\alpha s_r) \; ; \quad s_r > 0. \tag{2.4}$$

From (2.2) and (2.4), the marginal pdf of S_r is

$$f(s_r) = \frac{\mu^v s_r^{r-1}}{\Gamma(r)\,\Gamma(v)} \int_0^\infty \alpha^{r+v-1} exp\{-\alpha(s_r + \mu)\} d\alpha,$$

or,

$$f(s_r) = \frac{\mu^v s_r^{r-1}}{B(r,v)(s_r+\mu)^{r+v}} \;;\; s_r > 0. \tag{2.5}$$

Denoting by $\hat{\theta}_{BG}$ *and* $L(\hat{\theta}_{BG}, \theta)$, the Bayes estimators of $\theta = \psi(\alpha)$, under GELF and the loss resulting from estimating θ *by* $\hat{\theta}_{BG}$, respectively, the associated risk is defined by

$$R_G(\hat{\theta}_{BG}) = E_{S_r|\alpha}\left\{ L(\hat{\theta}_{BG}, \theta)\right\}. \tag{2.6}$$

The posterior risk for estimating θ *by* $\hat{\theta}_{BG}$ is

$$R_{PG}(\hat{\theta}_{BG}) = E_{\alpha|S_r}\left\{ L(\hat{\theta}_{BG}, \theta)\right\}, \tag{2.7}$$

and, Bayes risk for estimating θ *by* $\hat{\theta}_{BG}$ is

$$R_{BG}(\hat{\theta}_{BG}) = E_{S_r}\left[E_{\alpha|S_r}\left\{ L(\hat{\theta}_{BG}, \theta)\right\}\right]. \tag{2.8}$$

It is to be noted here that the risk of Bayes estimator of θ is a function of the modal parameter α and is independent of the sample data, the posterior risk is a function of the sample data S_r and of the prior parameters, and is independent of α, and Bayes risk is a function of the prior parameters only. We also note the following relationships

$$R_{BG}(\hat{\theta}_{BG}) = E_{\alpha}\left[\left\{ R_G(\hat{\theta}_{BG})\right\}\right],$$

and

$$R_{BG}(\hat{\theta}_{BG}) = E_{S_r}\left[\left\{ R_{PG}(\hat{\theta}_{BG})\right\}\right].$$

If $\hat{\theta}$ be the estimator of θ, then GELF is given by

$$L(\hat{\theta}, \theta) = (\hat{\theta}/\theta)^a - a\ln(\hat{\theta}/\theta) - 1;\; a \neq 0. \tag{2.9}$$

Under GELF, Bayes estimators of θ is given by

$$\hat{\alpha}_{BG} = \left[E_{\alpha/S_r}\left\{\alpha^{-a}\right\}\right]^{-1/a}. \tag{2.10}$$

In what follows, we obtain $\hat{\alpha}_{BG}$ and it's various measures of performance under GELF.

3. Bayes Estimators Of The Powers Of α, R(t) And 'P' Under Gelf

The following theorem provides Bayes estimators of powers of α.

Theorem 1: *For a positive integer p, under GELF, Bayes estimators of* α^{p} and $\alpha^{-\mathrm{p}}$ *are given, respectively, by* $\hat{\alpha}_{BG}^{p}$ *and* $\hat{\alpha}_{BG}^{-p}$ *where*

$$\hat{\alpha}_{\mathrm{BG}}^{\mathrm{p}} = \left\{ \frac{\Gamma(\mathrm{r}+\nu)}{\Gamma(\mathrm{r}+\nu\text{-ap})} \right\}^{1/a} (s_{\mathrm{r}}+\mu)^{-\mathrm{p}} ; ap < r+\nu, \tag{3.1}$$

and,

$$\hat{\alpha}_{\mathrm{BG}}^{\text{-p}} = \left\{ \frac{\Gamma(\mathrm{r}+\nu)}{\Gamma(\mathrm{r}+\nu\text{+ap})} \right\}^{1/a} (s_{\mathrm{r}}+\mu)^{\mathrm{p}} . \tag{3.2}$$

Proof: From (2.10),

$$\hat{\alpha}_{BG}^{p} = \left[E_{\alpha/S_r} \left\{ \left(\alpha^{p}\right)^{-a} \right\} \right]^{-1/a}$$

Using (2.3), we get

$$\mathrm{E}_{\alpha/\mathrm{S_r}}\left(\alpha^{\text{-ap}}\right) = \int_0^\infty \frac{(s_{\mathrm{r}}+\mu)^{\mathrm{r}+\nu}}{\Gamma(\mathrm{r}+\nu)} \alpha^{\text{-ap}} \alpha^{\mathrm{r}+\nu-1} \exp\left\{-\alpha(s_{\mathrm{r}}+\mu)\right\} \mathrm{d}\alpha,$$

or,

$$\mathrm{E}_{\alpha/\mathrm{S_r}}\left(\alpha^{\text{-ap}}\right) = \frac{(s_{\mathrm{r}}+\mu)^{\mathrm{r}+\nu}}{\Gamma(\mathrm{r}+\nu)} \int_0^\infty \frac{\mathrm{y}^{\mathrm{r}+\nu\text{-ap}-1}}{(s_{\mathrm{r}}+\mu)^{\mathrm{r}+\nu-\mathrm{ap}}} \exp(-\mathrm{y}) \mathrm{dy},$$

or,

$$\mathrm{E}_{\alpha/\mathrm{S_r}}\left(\alpha^{\text{-ap}}\right) = \frac{\Gamma(\mathrm{r}+\nu-\mathrm{ap})}{\Gamma(\mathrm{r}+\nu)} (s_{\mathrm{r}}+\mu)^{\mathrm{p}} ;\mathrm{ap}<\mathrm{r}+\nu. \tag{3.3}$$

Result (3.1) follows from (2.10) and (3.3). The proof of result (3.2) is similar to the proof of result (3.1).

In the following theorem, we derive expressions for the risks, posterior risks and Bayes risks of Bayes estimators of powers of α.

Theorem 2:

$$\mathrm{R_G}\left(\hat{\alpha}_{\mathrm{BG}}^{\mathrm{p}}\right) = \left\{ \frac{\Gamma(\mathrm{r}+\nu)}{\Gamma(\mathrm{r}+\nu-\mathrm{ap})} \right\} \frac{1}{\Gamma(\mathrm{r})} \int_0^\infty (\mathrm{z} + \alpha\mu)^{\text{-ap}} \mathrm{z}^{\mathrm{r}-1} \exp(-\mathrm{z}) \mathrm{dz}\text{-}\ln\left\{ \frac{\Gamma(\mathrm{r}+\nu)}{\Gamma(\mathrm{r}+\nu-\mathrm{ap})} \right\}$$
$$+ \frac{\mathrm{ap}}{\Gamma(\mathrm{r})} \int_0^\infty \ln(\mathrm{z} + \alpha\mu) \mathrm{z}^{\mathrm{r}-1} \exp(-\mathrm{z}) \mathrm{dz} - 1 ;\mathrm{r}+\nu>\mathrm{ap}, \tag{3.4}$$

$$\mathrm{R_{PG}}\left(\hat{\alpha}_{\mathrm{BG}}^{\mathrm{p}}\right) = \mathrm{ap}\,\psi(\mathrm{r}+\nu)\text{-}\ln\left\{ \frac{\Gamma(\mathrm{r}+\nu)}{\Gamma(\mathrm{r}+\nu-\mathrm{ap})} \right\} ;\mathrm{r}+\nu>\mathrm{ap}, \tag{3.5}$$

$$R_{BG}\left(\hat{\alpha}^{p}_{BG}\right)=ap\psi(r+\nu)\text{-}\ln\left\{\frac{\Gamma(r+\nu)}{\Gamma(r+\nu-ap)}\right\};r+\nu>ap, \tag{3.6}$$

$$R_{G}\left(\hat{\alpha}^{-p}_{BG}\right)=\left\{\frac{\Gamma(r+\nu)}{\Gamma(r+\nu+ap)}\right\}\frac{1}{\Gamma(r)}\int_0^\infty(z+\alpha\mu)^{ap}z^{r-1}\exp(-z)\,dz\text{-}\ln\left\{\frac{\Gamma(r+\nu)}{\Gamma(r+\nu+ap)}\right\}$$
$$-\frac{ap}{\Gamma(r)}\int_0^\infty\ln(z+\alpha\mu)z^{r-1}\exp(-z)\,dz-1, \tag{3.7}$$

$$R_{PG}\left(\hat{\alpha}^{-p}_{BG}\right)=-\left[ap\psi(r+\nu)+\ln\left\{\frac{\Gamma(r+\nu)}{\Gamma(r+\nu+ap)}\right\}\right], \tag{3.8}$$

and,

$$R_{BG}\left(\hat{\alpha}^{-p}_{BG}\right)=-\left[ap\psi(r+\nu)+\ln\left\{\frac{\Gamma(r+\nu)}{\Gamma(r+\nu+ap)}\right\}\right]. \tag{3.9}$$

Proof. From (2.9) and (3.1)

$$R_G\left(\hat{\alpha}^p_{BG}\right)=E_{S_r/\alpha}\left[\left(\hat{\alpha}^p_{BG}/\alpha^p\right)^a-a\ln\left(\hat{\alpha}^p_{BG}/\alpha^p\right)-1\right],$$

or,

$$R_G\left(\hat{\alpha}^p_{BG}\right)=E_{S_r/\alpha}\left[\left\{\frac{\Gamma(r+\nu)}{\Gamma(r+\nu-ap)}\right\}\left\{\frac{(s_r+\mu)^{-p}}{\alpha^p}\right\}^a-\ln\left\{\frac{\Gamma(r+\nu)}{\Gamma(r+\nu-ap)}\right\}^{1/a}\left\{\frac{(s_r+\mu)^{-p}}{\alpha^p}\right\}-1\right],$$

or,

$$R_G\left(\hat{\alpha}^p_{BG}\right)=\left\{\frac{\Gamma(r+\nu)}{\Gamma(r+\nu-ap)}\right\}E_{S_r/\alpha}\left[\left\{\alpha(s_r+\mu)^{-ap}\right\}\right]-\ln\left\{\frac{\Gamma(r+\nu)}{\Gamma(r+\nu-ap)}\right\}$$
$$+apE_{S_r/\alpha}\ln\left[\alpha(s_r+\mu)\right]-1.$$

Hence, result (3.4) follows on using (2.4), for $\alpha S_r=Z$.

From (2.9),

$$R_{PG}\left(\hat{\alpha}^p_{BG}\right)=E_{\alpha/S_r}\left[\left(\hat{\alpha}^p_{BG}/\alpha^p\right)^a-a\ln\left(\hat{\alpha}^p_{BG}/\alpha^p\right)-1\right],$$

or,

$$\mathrm{R}_{PG}\left(\hat{\alpha}_{BG}^{p}\right)=\left\{\frac{\Gamma(r+\nu)}{\Gamma(r+\nu-ap)}\right\}\mathrm{E}_{\alpha/\mathrm{S_r}}\left[\left\{\alpha\left(s_{\mathrm{r}}+\mu\right)\right\}^{-ap}\right]-\ln\left\{\frac{\Gamma(r+\nu)}{\Gamma(r+\nu-ap)}\right\}$$
$$+ap\ln\left(s_{\mathrm{r}}+\mu\right)+ap\mathrm{E}_{\alpha/\mathrm{S_r}}\left(\ln\alpha\right)-1. \tag{3.10}$$

From (2.3),

$$\mathrm{E}_{\alpha/\mathrm{S_r}}\left(\ln\alpha\right)=\frac{\left(s_{\mathrm{r}}+\mu\right)^{r+\nu}}{\Gamma(r+\nu)}\int_{o}^{\infty}\left(\ln\alpha\right)\alpha^{r+\nu-1}\exp\left\{-\left(s_{\mathrm{r}}+\mu\right)\alpha\right\}d\alpha,$$

Utilizing a result of Gradshteyn and Ryzhik (1980, p.286 Section 3.197(3)) that

$$\int_{o}^{\infty}x^{\nu-1}\exp(-\mu x)\,dx=\frac{\Gamma(r)}{\mu^{\nu}}\left[\psi(\nu)-\ln\mu\right];\quad \text{where, } \psi(x)=\frac{d}{dx}\Gamma(x),$$

we have

$$\mathrm{E}_{\alpha/\mathrm{S_r}}\left(\ln\alpha\right)=\left[\psi(r+\nu)-\ln\left(s_{\mathrm{r}}+\mu\right)\right]. \tag{3.11}$$

Now,

$$\mathrm{E}_{\alpha/\mathrm{S_r}}\left[\left\{\alpha\left(s_{\mathrm{r}}+\mu\right)\right\}^{-ap}\right]=\frac{\left(s_{\mathrm{r}}+\mu\right)^{r+\nu}}{\Gamma(r+\nu)}\int_{0}^{\infty}\alpha^{r+\nu-ap-1}\exp\left\{-\alpha\left(s_{\mathrm{r}}+\mu\right)\right\}d\alpha,$$

or,

$$\mathrm{E}_{\alpha/\mathrm{S_r}}\left[\left\{\alpha\left(s_{\mathrm{r}}+\mu\right)\right\}^{-ap}\right]=\frac{\left(s_{\mathrm{r}}+\mu\right)^{ap}}{\Gamma(r+\nu)}\Gamma(r+\nu-ap);r+\nu>ap, \tag{3.12}$$

hence, result (3.5) follows on combining (3.10), (3.11) and (3.12). Since, $\mathrm{R}_{\mathrm{PG}}\left(\hat{\alpha}_{\mathrm{BG}}^{\mathrm{p}}\right)$, is free from $\mathrm{S_r}$, the expression (3.6), of Bayes risk of $\left(\hat{\alpha}_{BG}^{p}\right)$, is similar to posterior risk.

The proof of the results (3.7), (3.8) and (3.9) are similar to those of (3.4), (3.5) and (3.6), respectively.

In the following lemma, we obtain Bayes estimator of the cdf, given at (1.1) at a specified point ‘x’ with the help of Bayes estimators of powers of α.

Lemma 1: *For* $F(\mathrm{x};\alpha,\lambda,\delta)$*, defined at (1.1)*

$$\hat{\mathrm{F}}_{BG}(\mathrm{x};\alpha,\lambda,\delta)=\sum_{i=0}^{\infty}\frac{1}{i!}\left\{\frac{\Gamma(r+\nu)}{\Gamma(r+\nu-ai)}\right\}^{1/a}\left[\frac{\ln\left(1-\mathrm{e}^{-\lambda\mathrm{x}^{\delta}}\right)}{(s_r+\mu)}\right]^{i}\quad ;\mathrm{ai}<r+\nu\ ;\mathrm{x},\alpha,\lambda,\delta>0. \qquad (3.13)$$

Proof. $F(\mathrm{x};\alpha,\lambda,\delta)$, given at (1.1), can be written as

$$F(\mathrm{x};\alpha,\lambda,\delta)=\exp\left[\alpha\ln\left(1-e^{-\lambda x^{\delta}}\right)\right],$$

or,

$$\mathrm{F}(\mathrm{x};\alpha,\lambda,\delta)=\sum_{i=0}^{\infty}\frac{\left[\ln\left(1-e^{-\lambda x^{\delta}}\right)\right]^{i}}{i!}\alpha^{i}.$$

Utilizing Lemma 1, of Chaturvedi and Tomar (2002), we get

$$\hat{\mathrm{F}}_{BG}(\mathrm{x};\alpha,\lambda,\delta)=\sum_{i=0}^{\infty}\frac{\left[\ln\left(1-e^{-\lambda x^{\delta}}\right)\right]^{i}}{i!}\hat{\alpha}_{BG}{}^{i},$$

therefore, Lemma 1 follows on using (3.1).

Corollary 1: *In particular, for a=1*

$$\hat{\mathrm{F}}_{BG}(\mathrm{x};\alpha,\lambda,\delta)=\left[1+\frac{\ln\left(1-\mathrm{e}^{-\lambda\mathrm{x}^{\delta}}\right)}{(s_r+\mu)}\right]^{r+\nu-1}\quad ;\mathrm{x},\alpha,\lambda,\delta>0. \qquad (C1)$$

Proof: Putting a=1, in (3.13), we have

$$\hat{\mathrm{F}}_{BG}(\mathrm{x};\alpha,\lambda,\delta)=\sum_{i=0}^{\infty}\frac{1}{i!}\left\{\frac{\Gamma(r+\nu)}{\Gamma(r+\nu-i)}\right\}\left[\frac{\ln\left(1-\mathrm{e}^{-\lambda\mathrm{x}^{\delta}}\right)}{(s_r+\mu)}\right]^{i},$$

or,

$$\hat{\mathrm{F}}_{BG}(\mathrm{x};\alpha,\lambda,\delta)=\sum_{i=0}^{\infty}\frac{1}{i!}\frac{(r+\nu-1)!}{(r+\nu-i-1)!}\left[\frac{\ln\left(1-\mathrm{e}^{-\lambda\mathrm{x}^{\delta}}\right)}{(s_r+\mu)}\right]^{i},$$

or,

$$\hat{\mathrm{F}}_{BG}\left(\mathrm{x};\alpha,\lambda,\delta\right)=\sum_{i=0}^{(r+\nu-1)}\binom{(r+\nu-1)}{i}\left[\frac{\ln\left(1-\mathrm{e}^{-\lambda x^{\delta}}\right)}{\left(s_r+\mu\right)}\right]^i,$$

hence, the corollary follows.

In the following Lemma, we obtain Bayes estimator of the pdf, given at (1.1) at a specified point ‘x’.

Lemma 2: *For* $f\left(\mathrm{x};\alpha,\lambda,\delta\right)$*, defined at (1.1)*

$$\hat{\mathrm{f}}_{BG}\left(\mathrm{x};\alpha,\lambda,\delta\right)=\frac{\lambda\delta x^{\delta-1}e^{-\lambda x^{\delta}}}{\left(s_r+\mu\right)\left(1-e^{-\lambda x^{\delta}}\right)}\sum_{i=0}^{\infty}\frac{1}{i!}\left\{\frac{\Gamma\left(r+\nu\right)}{\Gamma\left(r+\nu-a\left(i+1\right)\right)}\right\}^{1/a}\left[\frac{\ln\left(1-e^{-\lambda x^{\delta}}\right)}{\left(s_r+\mu\right)}\right]^i;a(i+1)<r+\nu.\qquad(3.14)$$

Proof: Lemma follows from the fact that

$$\frac{d}{dx}\hat{\mathrm{F}}_{BG}\left(\mathrm{x};\alpha,\lambda,\delta\right)=\hat{f}_{BG}\left(\mathrm{x};\alpha,\lambda,\delta\right).$$

Corollary 2: *In particular***,** *for a=1*

$$\hat{\mathrm{f}}_{BG}\left(\mathrm{x};\alpha,\lambda,\delta\right)=\frac{\left(r+\nu-1\right)\lambda\delta x^{\delta-1}e^{-\lambda x^{\delta}}}{\left(s_r+\mu\right)\left(1-e^{-\lambda x^{\delta}}\right)}\left[1+\frac{\ln\left(1-e^{-\lambda x^{\delta}}\right)}{\left(s_r+\mu\right)}\right]^{r+\nu-2}.\qquad(C2)$$

Proof: From (3.14), for a=1, we have

$$\hat{\mathrm{f}}_{BG}\left(\mathrm{x};\alpha,\lambda,\delta\right)=\frac{\lambda\delta x^{\delta-1}e^{-\lambda x^{\delta}}}{\left(s_r+\mu\right)\left(1-e^{-\lambda x^{\delta}}\right)}\sum_{i=0}^{\infty}\frac{1}{i!}\left\{\frac{\Gamma\left(r+\nu\right)}{\Gamma\left(r+\nu-i-1\right)}\right\}\left[\frac{\ln\left(1-e^{-\lambda x^{\delta}}\right)}{\left(s_r+\mu\right)}\right]^i,$$

or,

$$\hat{\mathrm{f}}_{BG}\left(\mathrm{x};\alpha,\lambda,\delta\right)=\frac{\left(r+\nu-1\right)\lambda\delta x^{\delta-1}e^{-\lambda x^{\delta}}}{\left(s_r+\mu\right)\left(1-e^{-\lambda x^{\delta}}\right)}\sum_{i=0}^{r+\nu-2}\binom{r+\nu-2}{i}\left[\frac{\ln\left(1-e^{-\lambda x^{\delta}}\right)}{\left(s_r+\mu\right)}\right]^i,$$

or,

$$\hat{\mathrm{f}}_{BG}\left(\mathrm{x};\alpha,\lambda,\delta\right)=\frac{\left(r+\nu-1\right)\lambda\delta x^{\delta-1}e^{-\lambda x^{\delta}}}{\left(s_r+\mu\right)\left(1-e^{-\lambda x^{\delta}}\right)}\left[1+\frac{\ln\left(1-e^{-\lambda x^{\delta}}\right)}{\left(s_r+\mu\right)}\right]^{r+\nu-2},$$

and the corollary fallows.

In the following Theorem, we obtain Bayes estimator of the reliability function.

Theorem 3: *For the reliability function R(t), given at (1.1)*

$$\hat{\mathrm{R}}_{BG}(\mathrm{t})=1-\sum_{i=0}^{\infty}\frac{1}{i!}\left\{\frac{\Gamma(r+\nu)}{\Gamma\left(r+\nu-ai\right)}\right\}^{1/a}\left[\frac{\ln\left(1-\mathrm{e}^{-\lambda\mathrm{t}^{\delta}}\right)}{\left(s_r+\mu\right)}\right]^{i};ai<r+\nu. \tag{3.15}$$

Proof. Result follows from the fact that

$$\hat{R}_{BG}\left(t\right)=1-\hat{F}_{BG}\left(\mathrm{t};\alpha,\lambda,\delta\right).$$

Corollary 3: *In particular, for a=1*

$$\hat{\mathrm{R}}_{BG}(\mathrm{t})=1-\left[1+\frac{\ln\left(1-\mathrm{e}^{-\lambda\mathrm{t}^{\delta}}\right)}{\left(s_r+\mu\right)}\right]^{r+\nu-1}. \tag{C3}$$

Proof. From the argument similar to used in theorem 3, and from (C2), corollary follows.

In the following theorem, we obtain Bayes estimator of P.

Theorem 4:

$$\hat{\mathrm{P}}_{BG}=\int_{y=0}^{\infty}\left[1-\sum_{i=0}^{\infty}\frac{1}{i!}\left\{\frac{\Gamma(r+\nu_1)}{\Gamma\left(r+\nu_1-ai\right)}\right\}^{1/a}\left\{\frac{\ln\left(1-e^{-\lambda_1 y^{\delta_1}}\right)}{\left(s_r+\mu_1\right)}\right\}^{i}\right]$$
$$\times\left[\frac{\lambda_2\delta_2 y^{\delta_2-1}e^{-\lambda_2 y^{\delta_2}}}{\left(t_s+\mu_2\right)\left(1-e^{-\lambda_2 y^{\delta_2}}\right)}\sum_{j=0}^{\infty}\frac{1}{j!}\left\{\frac{\Gamma(s+\nu_2)}{\Gamma\left(s+\nu_2-a(j+1)\right)}\right\}^{1/a}\left\{\frac{\ln\left(1-e^{-\lambda_2 y^{\delta 2}}\right)}{\left(t_s+\mu_2\right)}\right\}^{i}\right]dy \tag{3.16}$$

$$;r+\nu_1>ai,\ \ and,\ \ s+\nu_2>a(j+1).$$

Proof. We know that

$$\hat{P}_{BG} = \int\limits_{y=0}^{\infty} \int\limits_{x=y}^{\infty} \hat{f}_{BG}\left(x;\alpha_1,\lambda_1,\delta_1\right) \hat{f}_{BG}\left(y;\alpha_2,\lambda_2,\delta_2\right) dxdy,$$

or,

$$\hat{P}_{BG} = \int\limits_{y=0}^{\infty} \hat{R}_{BG}\left(y;\alpha_1,\lambda_1,\delta_1\right) \hat{f}_{BG}\left(y;\alpha_2,\lambda_2,\delta_2\right) dy. \tag{3.17}$$

Result follows on combining Lemma 2, Theorem 3 and (3.17).

Corollary 4: *In particular, for a=1*

$$\hat{\mathrm{P}}_{BG} = 1 - \frac{(s+\nu_2-1)\lambda_2\delta 2}{\left(t_s+\mu_2\right)} \int\limits_{y=0}^{\infty} \frac{y^{\delta_2-1}e^{-\lambda_2 y^{\delta_2}}}{\left(1-e^{-\lambda_2 y^{\delta_2}}\right)} \left\{1+\frac{\ln\left(1-e^{-\lambda_1 y^{\delta_1}}\right)}{\left(s_r+\mu_1\right)}\right\}^{r+\nu_1-1} \times \left\{1+\frac{\ln\left(1-e^{-\lambda_2 y^{\delta 2}}\right)}{\left(t_s+\mu_2\right)}\right\}^{s+\nu_2-2} dy. \tag{C4}$$

Proof. From (3.16) and for a=1, we have

$$\hat{\mathrm{P}}_{BG} = \int\limits_{y=0}^{\infty} \left[1-\left\{1+\frac{\ln\left(1-e^{-\lambda_1 y^{\delta_1}}\right)}{\left(s_r+\mu_1\right)}\right\}^{r+\nu_1-1}\right] \frac{(s+\nu_2-1)\lambda_2\delta_2 y^{\delta_2-1}e^{-\lambda_2 y^{\delta_2}}}{\left(t_s+\mu_2\right)\left(1-e^{-\lambda_2 y^{\delta_2}}\right)} \left\{1+\frac{\ln\left(1-e^{-\lambda_2 y^{\delta 2}}\right)}{\left(t_s+\mu_2\right)}\right\}^{s+\nu_2-2} dy,$$

or,

$$\hat{\mathrm{P}}_{BG} = \frac{(s+\nu_2-1)\lambda_2\delta_2}{\left(t_s+\mu_2\right)} \int\limits_{y=0}^{\infty} \frac{y^{\delta_2-1}e^{-\lambda_2 y^{\delta_2}}}{\left(1-e^{-\lambda_2 y^{\delta_2}}\right)} \left\{1+\frac{\ln\left(1-e^{-\lambda_2 y^{\delta_2}}\right)}{\left(t_s+\mu_2\right)}\right\}^{s+\nu_2-2} dy$$
$$- \int\limits_{y=0}^{\infty} \frac{(s+\nu_2-1)\lambda_2\delta_2 y^{\delta_2-1}e^{-\lambda_2 y^{\delta_2}}}{\left(t_s+\mu_2\right)\left(1-e^{-\lambda_2 y^{\delta_2}}\right)} \left[1-\left\{1+\frac{\ln\left(1-e^{-\lambda_1 y^{\delta_1}}\right)}{\left(s_r+\mu_1\right)}\right\}^{r+\nu_1-1}\right] \left\{1+\frac{\ln\left(1-e^{-\lambda_2 y^{\delta_2}}\right)}{\left(t_s+\mu_2\right)}\right\}^{s+\nu_2-2} dy \tag{3.18}$$

Let,

$$I=\frac{(s+\nu_2-1)\lambda_2\delta_2}{\left(t_s+\mu_2\right)}\int_{y=0}^{\infty}\frac{y^{\delta_2-1}e^{-\lambda_2 y^{\delta_2}}}{\left(1-e^{-\lambda_2 y^{\delta_2}}\right)}\left\{1+\frac{\ln\left(1-e^{-\lambda_2 y^{\delta_2}}\right)}{\left(t_s+\mu_2\right)}\right\}^{s+\nu_2-2}dy,$$

putting, $1+\frac{\ln\left(1-e^{-\lambda_2 Y^{\delta_2}}\right)}{\left(T_s+\mu_2\right)}=\frac{1}{Z}$, We have, I=1 and (3.18) becomes

$$\hat{\mathrm{P}}_{BG}=1-\frac{(s+\nu_2-1)\lambda_2\delta 2}{\left(t_s+\mu_2\right)}\int_{y=0}^{\infty}\frac{y^{\delta_2-1}e^{-\lambda_2 y^{\delta_2}}}{\left(1-e^{-\lambda_2 y^{\delta_2}}\right)}\left\{1+\frac{\ln\left(1-e^{-\lambda_1 y^{\delta_1}}\right)}{\left(s_r+\mu_1\right)}\right\}^{r+\nu_1-1}$$
$$\times\left\{1+\frac{\ln\left(1-e^{-\lambda_2 y^{\delta 2}}\right)}{\left(t_s+\mu_2\right)}\right\}^{s+\nu_2-2}dy,$$

which is the required result.

REMARKS

I. It is interesting to note that the expression for $\mathrm{R}_{\mathrm{PG}}\left(\hat{\alpha}_{\mathrm{BG}}^{\mathrm{p}}\right)$ *and* $\mathrm{R}_{\mathrm{BG}}\left(\hat{\alpha}_{\mathrm{BG}}^{\mathrm{p}}\right)$ $\left[\mathrm{R}_{\mathrm{PG}}\left(\hat{\alpha}_{\mathrm{BG}}^{\mathrm{-p}}\right)\ \textit{and}\ \mathrm{R}_{\mathrm{PG}}\left(\hat{\alpha}_{\mathrm{BG}}^{\mathrm{-p}}\right)\right]$ are same.

II. Complete sample case can be obtained on putting, r = n and s = m.

III. For, a = -1, $\hat{\alpha}_{BG}^{p}$ coincide with $\hat{\alpha}_{BS}^{p}$ [Bayes estimator of α^{p} under squared-error loss function (SELF)].

IV. In order to obtain Bayes estimators of R(t) and ‘P’, in the literature, authors first obtain the expression of these quantities and then their Bayes estimators, say, posterior mean under GELF. In the present approach to obtain Bayes estimators of R(t) and ‘P’, we made use of Bayes estimators of the cdf/pdf and one does not need their expressions.

V. We have established an interrelationship between the estimation of R(t) and ‘P’.

VI. From (3.2),

$$\hat{\alpha}_{BG}^{-1} = \frac{\Gamma(r+v)}{\Gamma(r+v+1)}(s_r+\mu) = \frac{(s_r+\mu)}{(r+v)} ,$$

for, a = 1

$$E(\hat{\alpha}_{BG}^{-1}) = \frac{E(s_r)+\mu}{(r+v)} = \frac{\frac{1}{2\alpha}E(\chi^2_{2r})+\mu}{(r+v)} = \frac{\frac{r}{\alpha}+\mu}{(r+v)}$$

$$\to \alpha^{-1}, as, r \to \infty.$$

Moreover,

$$V(\hat{\alpha}_{BG}^{-1}) = \frac{V(s_r)}{4\alpha^2(r+v)^2} = \frac{r}{\alpha^2(r+v)^2} \to 0, as, r \to \infty.$$

Therefore, $\hat{\alpha}_{BG}^{-1}$ is a consistent estimator of α^{-1}.

Now, (C2) can be written as

$$\hat{f}_{BG}(x;\alpha,\lambda,\delta) = \frac{(r+v-1)\lambda\delta x^{\delta-1}e^{-\lambda x^\delta}}{(s_r+\mu)(1-e^{-\lambda x^\delta})}\left[1+\frac{ln(1-e^{-\lambda x^\delta})}{(s_r+\mu)}\right]^{(r+v-2)} ,$$

or,

$$\hat{f}_{BG}(x;\alpha,\lambda,\delta) = \frac{(r+v-1)\lambda\delta x^{\delta-1}e^{-\lambda x^\delta}}{(r+v)\hat{\alpha}_{BG}^{-1}(1-e^{-\lambda x^\delta})}\left[1+\frac{ln(1-e^{-\lambda x^\delta})}{(r+v)\hat{\alpha}_{BG}^{-1}}\right]^{(r+v-2)}$$

$$\to \frac{\alpha\lambda\delta x^{\delta-1}e^{-\lambda x^\delta}}{(1-e^{-\lambda x^\delta})}e^{\alpha\, ln(1-e^{-\lambda x^\delta})} = f(x;\alpha,\lambda,\delta), as, r \to \infty.$$

Hence, $\hat{f}_{BG}(x;\alpha,\lambda,\delta)$ is a consistent estimator of $f(x;\alpha,\lambda,\delta)$. Similarly we can prove that $\hat{R}(t)_{BG}$ and, $\hat{P}_{BG}$ are also consistent estimators of R(t) and 'P', respectively.

VII. If we look at remark (VI), we observe that the estimator of negative powers of α is used to prove the consistency of $\hat{f}_{BG}(x;\alpha,\lambda,\delta)$. This justifies the estimation of negative powers of α.

4. Simulation Studies

In order to validate the performance of the estimators under GELF with respect to the actual reliability estimate obtained through numerical integration, we have simulated a sample (0.9437297, 1.3012949, 1.2275004, 1.3431583, 1.1655092, 1.3982024, 1.1047822, 0.9083479, 0.9530192, 1.4253371) of size 10 from (1.1), with α

= 2, λ = 1 and δ = 3.5. For prior parameters μ = 5, ν = 6, we get S_r = 2.651203. In Fig.1, we have plotted prior and posterior densities. For p=1, we obtained table 1.

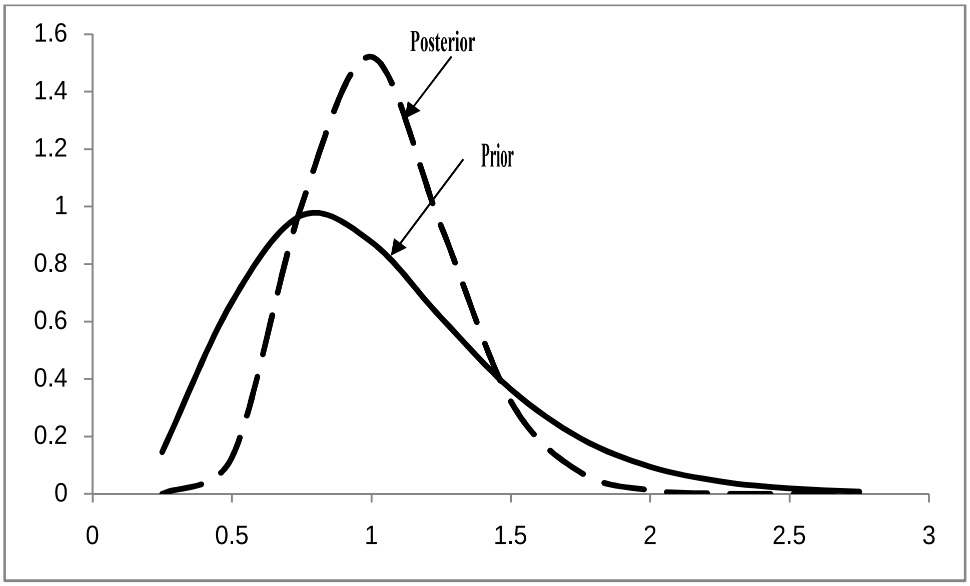

Fig. 1: Prior and Posterior densities.

Table 1.

$\hat{R}_{BG}(0.85) = 0.8247097$	
$\hat{\alpha}_{BG}$ = 1.960476	$\hat{\alpha}^{-1}_{BG}$ = 0.4782002
$R_{PG}(\hat{\alpha}_{BG}) = R_{BG}(\hat{\alpha}_{BG})$ = 0.03296313	
$R_{PG}(\hat{\alpha}^{-1}_{BG}) = R_{BG}(\hat{\alpha}^{-1}_{BG})$ =0.03157539	

In order to compute values of $R(t)$ (actual), $\hat{R}_{BG}(t)$. We have simulated a sample (0.8552144, 1.0913801, 0.8943219, 1.0151310, 0.7988348, 1.2434342, 1.3266559, 1.1189079, 1.0352386, 0.8403234, 1.4534861, 0.8598646, 0.9898228, 1.1965614, 0.8634094, 1.0089688, 1.0129357, 1.5252311, 1.5057447, 1.3545000) of size 20 from (1.1), with α=2, λ=1, and δ=3.5 and with the prior parameters μ=5, ν=7. In this case r = n = 20 and for a=1 we have S_r = 8.212747. Computed values of $R(t)$ (actual), $\hat{R}_{BG}(t)$, over t=0.30(0.10)1.10 are presented in Table 2.

Table 2.

T	$R(t)$	$\hat{R}_{BG}(t)$
0.30	0.9997845	0.9999549
0.40	0.9984264	0.9993116
0.50	0.9928437	0.9953943
0.60	0.9762639	0.9810994
0.70	0.9377652	0.9442345
0.80	0.8650034	0.8711252
0.90	0.7507785	0.7544591
1	0.6004236	0.6009298
1.10	0.4338854	0.4321579

For computing the value of $\hat{P}_{BG}$ we have simulated two samples of sizes r=30 and s=35 from (1.1), with parameters $(\alpha_1 = 3, \lambda_1 = 1, \delta_1 = 3.5)$ and $(\alpha_2 = 2.5, \lambda_2 = 1, \delta_2 = 2)$, respectively.

Sample 1:

1.0692569, 1.0611359, 1.3356502, 1.1388135, 0.9756152, 0.8198281, 0.9521456, 0.8827471, 1.1171857, 0.8436072, 1.3559319, 1.0899231, 1.4320338, 0.8501137, 0.8329285, 1.2574847, 1.2841388, 0.8687503, 1.0897072, 0.8405082, 1.1612826, 1.1422926, 1.2018069, 1.1394938, 1.3068323, 1.5348110, 1.3410500, 1.0625237, 0.6788137, 0.8629385.

Sample 2:

1.1093962, 1.5358428, 1.6884939, 1.4933483, 0.9063976, 0.9806163, 0.9516506, 0.5337585, 1.1117653, 1.1508700, 1.7651138, 1.1985941, 1.4516742, 1.2466224, 1.6064628, 1.2245379, 0.9929597, 0.5351120, 1.8114293, 1.5507258, 1.0665552, 1.9124794, 0.8617678, 1.0864834, 1.3286711, 0.5635906, 0.7426067, 0.4878706, 1.0646319, 0.4361158, 1.3022855, 0.7945857,0.6497159, 0.6605123, 0.5983352.

For the case when, prior parameters are $\left(\mu_1 = 5,\ \nu_1 = 5\right)$ and $\left(\mu_2 = 4,\ \nu_2 = 3\right)$, respectively, for a=1, we get S_r=12.72371, T_s=18.98478 and $\hat{P}_{BG}$=0.5142761.

In Fig. 2, we have plotted graph of $\hat{f}_{BG}(x;\alpha,\lambda,\delta)$ for different values of n=5(5)25 and 50 under GELF. We conclude from the figures that as n increases, the curve of $\hat{f}_{BG}(x;\alpha,\lambda,\delta)$ come close to the curve of $f(x;\alpha,\lambda,\delta)$. This justifies the consistency property of the estimators. In Fig. 3, we have plotted graph of $\hat{F}_{BG}(x;\alpha,\lambda,\delta)$ for different values of n=5(5)25 and 50 under GELF.

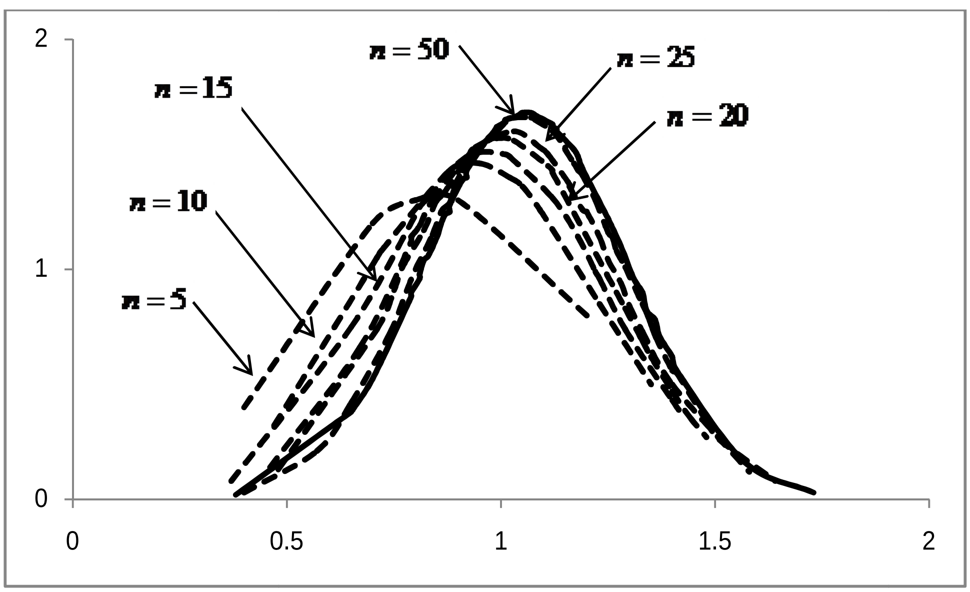

Fig. 2: The curves of $f(x;\alpha,\lambda,\delta)$ (bold) and $\hat{f}_{BG}(x;\alpha,\lambda,\delta)$ (dotted).

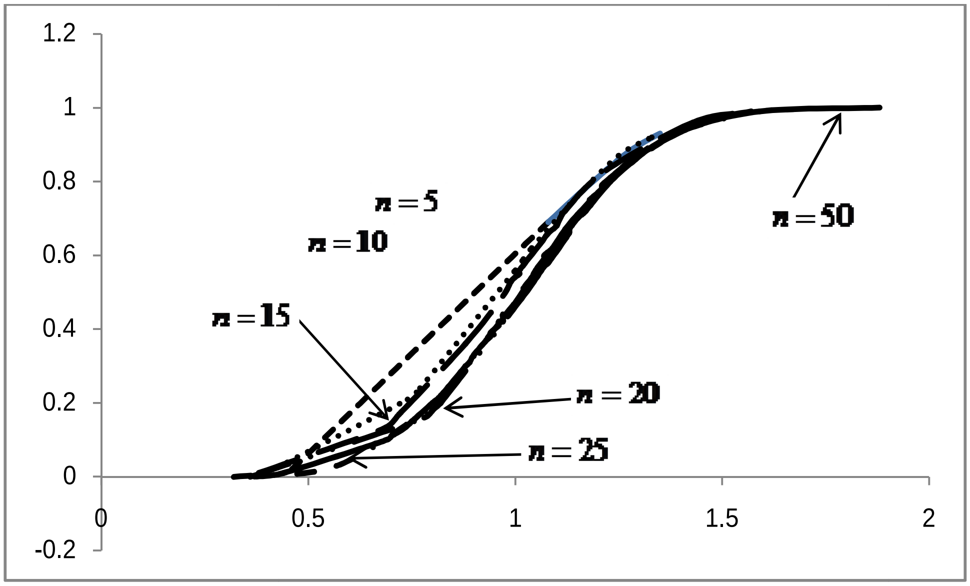

Fig. 3: The curves of $\hat{F}_{BG}(x;\alpha,\lambda,\delta)$ **.**

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