Parameter estimation of a process driven by fractional Brownian motion: An estimating function approach

Inderdeep Kaur, T. V. Ramanathan and **U. V. Naik-Nimbalkar**

Department of Statistics and Centre for Advanced Studies

University of Pune, Pune-411 007, India

Abstract

Statistical inference problems related to self similar processes and processes driven by the fractional Brownian motion have been studied extensively by various researchers, when the process is observed continuously. However, there were not enough attempts to investigate the inference problems of these processes, when the observational scheme is discrete in nature. Here we address this problem using the ideas of estimating functions. Optimal estimating function has been proposed for the estimation of the parameters of a process driven by fractional Brownian motion. Simulation studies incdicate that the computational time can be substantially reduced, if the proposed methods are used as opposed to approximations to the integrals that appears in the maximum likelihood estimator, under continuous observational scheme. We discuss the parameter estimation of fractional versions of Gompertzian and Vasicek models, using the proposed procedure.

Keywords: Discretely observed processes; Estimating equations; Fractional Brownian motions; Fractional Ornstein-Uhlenbeck type process; Self-similar processes.

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1 Introduction

Stochastic modeling is one of the important steps towards understanding a process which evolves over time. Autocovariances (or autocorrelations) plays a significant role in modeling such a processes. Long range dependence is a well known phenomenon, which occurs when the autocovariances of such a process exhibit a slow decay. Self similar processes is one such class of processes whose autocovariances decay slowly and thus related to the long range dependent processes. Self-similar processes are stochastic models with the property that a scaling in time is equivalent to an appropriate scaling in space. The connection between the two types of scaling is determined by a constant, known as *Hurst* exponent. For certain range of values of Hurst exponent, these process show long range dependence. Long memory processes are used for modeling various stochastic phenomena that arises in areas such as hydrology, geophysics, medicine, genetics and financial economics and more recently in modeling internet traffic patterns.

Fractional Brownian motion (fBm) is a special type of self-similar process. There have been some recent attempts to study the inference problems related to stochastic processes driven by a fBm for modeling a stochastic phenomena with possible long range dependence. In a series of papers Prakasa Rao (2003, 2004 and 2005) discussed various methods of estimation of the parameter of interest, including maximum likelihood estimation, in such processes. An extensive review on most of the recent developments related to the parametric and other inference procedures for such processes can be found in Prakasa Rao (2010).

Ideally, the inference concerning the parameters should be based on the likelihood of the process observed continuously. In practice, we often encounter discrete-time observations. For instance, Drakakis and Radulovic (2006) consider a self-similar model for the internet traffic by taking into account the fact that information transmission takes place (in quanta or data packets) discretely rather than continuously. The likelihood function for discrete observations is a product of transition densities, and these are completely known only in certain special cases. Therefore it is customary to use some kind of an approximation to the Maximum Likelihood Estimator (MLE) of a continuously observed process. However, such an estimator will not have many of the optimal properties. As an alternative to this, we develop estimating functions for the estimation of parameters of such processes.

The estimating function theory was originally introduced by Godambe (1960). It is a convenient way to handle the estimation problem by avoiding the distributional assumptions of the observed data. The method was further extended to stochastic processes by Godambe and Heyde (1987) by suggesting an optimality criteria for the estimating functions. Biby and Sorenson (1995) considered martingale estimating functions for discretely observed diffussion processes.

In this paper we discuss an estimating equation approach for the estimation of parameters of a discretely observed process which is governed by linear stochastic differential equations driven by fBm. In Section 2 we briefly review selfsimilar processes and some of its important properties. Details of linear SDE driven by fBm and maximum likelihood estimation (Prakasa Rao 2004) are reviewed briefly in Section 3. Further in this section, we suggest an estimating equation which is optimal in certain class, for the problem under consideration. In the next section, we restrict ourselves to fractional Ornstein Uhlenbeck process. Here we give discretized approximation of the MLE followed by a detailed discussion on estimating function. Exact formula is given for the estimator of parameter of interest θ. In Section 5, we report a simulation study, which compares the estimators from the proposed and maximum likelihood method. Two applications are considered in Section 6. Some concluding remarks form Section 7. All detailed mathematical derivations are deferred to Appendix A and B.

2 Long range dependence and Self-similarity

We start our discussion by defining the concepts of longrange dependence, self-similarity and fractional Brownian motion. Long range dependence is said to occur in a stationary time process $\{X_n, n \geq 0\}$ if the $Cov(X_0, X_n)$ of the process tends to zero as $n \to \infty$ and

$$\sum_{n=0}^{\infty} |Cov(X_0, X_n)| = \infty.$$

That is, the covariance between X_0 and X_n tends to zero slowly enough, so that their sums diverge.

A real-valued stochastic process $Z = \{Z_t, -\infty < t < \infty\}$ is said to be *self-similar* with index $H >$ if for any $a > 0$,

$$\mathcal{L}(\{Z_{at}, -\infty < t < \infty\}) = \mathcal{L}(\{a^H Z_t, -\infty < t < \infty\}) \tag{1}$$

where $\mathcal{L}$ denotes the class of all finite dimensional distributions and the equality indicates the equality of the finite dimensional distributions of the process on the right side of the equation (1) with the corresponding finite dimensional distributions of the process on the left side of the equation (1). The index H is called the *scaling exponent* or the *fractal index* or the *Hurst parameter* of the process. If H is the scaling exponent of a self-similar process Z, then the process Z is called a H*-self similar* process or H-ss process for short. Self-similarity refers to invariance in distribution under a suitable change of scale.

A Gaussian H-ss process $W^H = \{W_t^H, \infty < t < \infty\}$ with stationary increments and fractal index $0 < H < 1$ is called *fractional Brownian motion.* The fBm has following properties:

(i) If $H \neq 1$, then $E[W_t^H] = 0, \infty < t < \infty$,

(ii) $E[W_t^{H^2}] = |t|^{2H} E[W_1^{H^2}]$, and

(iii) $E[W_s^H W_t^H] = \frac{1}{2}\left[s^{2H} + t^{2H} - |s-t|^{2H}\right]$.

The process W^H is said to be standard fBm if $Var(W_t^H) = 1$. The sample paths of fBm are continuous with probability one but are not L^2 differentiable. The fBm reduces to Brownian motion for $H = \frac{1}{2}$. Every fBm is a H*-self similar* process and has stationary increments. For $H = 1$, the process becomes a straight line through the origin with random normal slope. Therefore interesting models are obtained with $0 < H < 1$ (cf. Samorodnitsky, 2006). For $1/2 < H < 1$ fractional Gaussian noise (increment process of the fBm) depicts long memory in its behavior. Therefore, fBm with $1/2 < H < 1$ have been used extensively to model various natural phenomena.

3 Linear SDE driven by fBm

Let $(\Omega, \mathcal{F}, (\mathcal{F})_t, P)$ be a stochastic basis satisfying the usual conditions. The natural filtration of a process is understood as the P-completion of the filtration generated by this process.

Let $W^H = \{W_t^H, t \geq 0\}$ be a normalized fractional Brownian motion (fBm) with known Hurst parameter $H \in (0,1)$. Let us consider a stochastic process $X = \{X_t, t \geq 0\}$ defined by the stochastic integral equation

$$X_t = \int_0^t [a(s, X_s) + \theta\ b(s, X_s)]ds + \int_0^t \sigma(s) dW_s^H = 0,\ t \geq 0,$$

where $\theta \in \Theta \subset R$, and $\sigma(t)$ is a positive nonvanishing function on $[0, \infty)$. Let,

$$C(\theta, t) = a(t, X_t) + \theta\ b(t, X_t),\ t \geq 0$$

and assume that the sample paths of the process $\left\{\frac{C(\theta,t)}{\sigma(s)}, t \geq 0\right\}$ are smooth enough. Here we note that the process X is not a semi-martingale. Therefore one can associate a semimartingale which is called a *fundamental semi martingale* such that the natural filtration of the semimartingale process coincides with the natural filtration of the process X (Kleptsyna et. al., 2000). Define the process $Q_t(\theta, H)$ as

$$Q_t(\theta, H) = \frac{d}{dw_t^H} \int_0^t \kappa_H(t,s) \frac{C(\theta, s)}{\sigma(s)} ds,\ t \geq 0,$$

where

$$\lambda_H = \frac{2H\Gamma(3-2H)\Gamma(H+\frac{1}{2})}{\Gamma(\frac{3}{2}-H)},$$

$$w_t^H = \lambda_H^{-1} t^{2-2H},$$

$$\kappa_H(t,s) = k_H^{-1} s^{\frac{1}{2}-H} (t-s)^{\frac{1}{2}-H},$$

and

$$k_H = 2H\Gamma(\frac{3}{2}-H)\Gamma(H+\frac{1}{2}),$$

for $0 < s < t$.

The derivative is understood in the sense of absolute continuity with respect to the measure generated by w^H. Define

$$Z_t = \int_0^t \frac{\kappa_H(t,s)}{\sigma(s)} dX_s,\ t \geq 0. \tag{2}$$

Then the process $Z = \{Z_t, t \geq 0\}$ is an $\mathcal{F}_t$-semimartingale with the decomposition

$$Z_t = \int_0^t Q_s(\theta, H) dw_s^H \; + \; M_t^H,$$

where M^H is the fundamental martingale defined by

$$M_t^H = \int_0^t \kappa_H(t,s) dW_s^H, \; t \geq 0.$$

3.1 Maximum likelihood estimation

Let P_θ^T be the measure induced by the process $\{X_t, 0 \leq t \leq T\}$. Then, the Radon-Nikodym derivative of P_θ^T with respect to P_0^T is given by

$$\frac{dP_\theta^T}{dP_0^T} = exp\left[\int_0^T Q_s(\theta, H) dZ_s - \frac{1}{2}\int_0^T Q_s^2(\theta, H) dw_s^H\right].$$

Suppose we observe the process continuously between $0 < t < T$. Then the continuous time log-likelihood function will be

$$\ell_T(\theta) = \int_0^T Q_s(\theta, H) dZ_s - \frac{1}{2}\int_0^T Q_s^2(\theta, H) dw_s^H. \tag{3}$$

Therefore the MLE $\hat{\theta}_T$ of θ is

$$\hat{\theta}_T = \frac{\int_0^T J_2(t) dZ_t + \int_0^T J_1(t) J_2(t) dw_t^H}{\int_0^T J_2^2(t) dw_t^H},$$

where

$$J_1(t) = \frac{d}{dw_t^H}\int_0^t \kappa_H(t,s)\frac{a(s, X_s)}{\sigma(s)} ds,$$

and

$$J_2(t) = \frac{d}{dw_t^H}\int_0^t \kappa_H(t,s)\frac{b(s, X_s)}{\sigma(s)} ds.$$

The MLE $\hat{\theta}_T$ is proved to be strongly consistent (Prakasa Rao, 2003). Limiting distribution of $\hat{\theta}$ is a mixture of the normal distributions with mean 0 and variance η^{-2} with the mixing distribution as that of η, where η is a random variable such that $P(\eta > 0) = 1$. Thus for computation of the MLE, one needs to compute these integrals.

3.2 Estimating equation

Now we look at the estimation of the parameter θ, from discretely observed data, through estimating equations. Suppose we observe the process $\{X_t : 0 \le t \le T\}$ at time points $\{u_1, u_2, \ldots, u_n\}$. For brevity we denote observation at time point u_t by X_t. Approximating the integrals in (3) with an Ito-sum and a Riemann sum,

$$\tilde{\ell_n}(\theta) = \sum_{i=2}^{n} Q_{i-1}(\theta, H)(Z_i - Z_{i-1}) - \frac{1}{2}\sum_{i=2}^{n} Q_{i-1}^2(\theta, H)(w_i^H - w_{i-1}^H).$$

Now differentiating with respect to θ, we get an approximate score function of the form

$$\dot{\tilde{\ell}}_n(\theta) = \sum_{i=2}^{n} \dot{Q}_{i-1}(\theta, H)(Z_i - Z_{i-1}) - \sum_{i=2}^{n} Q_{i-1}(\theta, H)\dot{Q}_{i-1}(\theta, H)(w_i^H - w_{i-1}^H), \tag{4}$$

where $\dot{Q}$ denotes the derivative of Q with respect to θ. Since the estimating function (4) is biased, we adjust it by subtracting the compensator to get a zero-mean martingale with respect to the filtration defined by $\mathcal{F}_i = \sigma(Z_1, \ldots, Z_i)$, $i = 1, 2, \ldots$. The compensator is

$$\sum_{i=2}^{n} E_\theta\{\dot{\tilde{\ell}}_i(\theta) - \dot{\tilde{\ell}}_{i-1}(\theta)|\mathcal{F}_{i-1}\} = \\ \sum_{i=2}^{n} \dot{Q}_{i-1}(\theta, H)\left\{F_{i-1}(\theta) - Z_{i-1}\right\} - \sum_{i=2}^{n} Q_{i-1}(\theta, H)\dot{Q}_{i-1}(\theta, H)(w_i^H - w_{i-1}^H),$$

where $F_{i-1}(\theta) = E_\theta\big[Z_i|\mathcal{F}_{i-1}\big]$. This leads to the following estimating function, which is a zero-mean martingale,

$$\tilde{G_n}(\theta) = \sum_{i=2}^{n} \dot{Q}_{i-1}(\theta, H)\big[Z_i - F_{i-1}(\theta)\big].$$

Now consider the class of estimating functions $\mathcal{G}$ such that $G_n \in \mathcal{G}$ if G_n is of the form

$$G_n(\theta) = \sum_{i=2}^{n} g_{i-1}(\theta)\big[Z_i - F_{i-1}(\theta)\big],$$

where g_{i-1} is $\mathcal{F}_{i-1}$-measurable and a continuously differentiable function of θ, $i = 2, \ldots, n$. Hence the optimal estimating function in the class $\mathcal{G}$ is given by

$$G_n^*(\theta) = \sum_{i=2}^{n} \frac{\dot{F}_{i-1}(\theta)}{\phi_{i-1}(\theta)}\big[Z_i - F_{i-1}(\theta)\big], \tag{5}$$

where

$$\phi_{i-1}(\theta) = E_\theta\left\{(Z_i - F_{i-1}(\theta))^2\,|\mathcal{F}_{i-1}\right\},$$

and

$$\dot{F}_{i-1}(\theta) = \frac{d}{d\theta} F_{i-1}(\theta).$$

To simplify the notational complexities we restrict ourselves to fractional Ornstein-Uhlenbeck type prosess which is also a linear SDE driven by a fBm

4 Fractional Ornstein-Uhlenbeck type process

Fractional Ornstein-Uhlenbeck type process is a fractional analogue of the Ornstein-Uhlenbeck process, that is, a continuous time first order autoregressive process $X = \{X_t, t \geq 0\}$ which is the solution of a one-dimensional homogeneous linear stochastic differential equation driven by a fBm $W^H = \{W_t^H, t \geq 0\}$ with Hurst parameter $H \in [1/2, 1)$. It is the unique Gaussian process satisfying the linear integral equation

$$X_t = \theta \int_0^t X_s ds + \sigma W_t^H, t \geq 0. \tag{6}$$

Note that using equation (3.7) of Kleptsyna (2002), process Q can also be represented as,

$$Q_t(H) = \frac{\lambda_H^*}{2} \Big\{ t^{2H-1} Z_t + \int_0^t r^{2H-1} dZ_r \Big\}, \tag{7}$$

where $\lambda_H^* = \frac{\lambda_H}{2(1-H)}$.

4.1 Discretization

Suppose we observe the process at time points $\{u_1, u_2, \ldots, u_n\}$. Discretization of (6) gives

$$\begin{aligned} X_1 &= 0 \\ X_2 &= \theta X_1 (u_2 - u_1) + \sigma W_2^H \\ &\vdots \\ X_t &= \theta \sum_{s=1}^{t-1} X_s (u_{s+1} - u_s) + \sigma W_t^H, \quad \text{for } t = 2, 3, \ldots, n. \end{aligned} \tag{8}$$

Similarly discretization of Z-process, given in (2), leads to

$$\begin{aligned} Z_1 &= 0 \\ Z_2 &= \frac{1}{\sigma}\kappa_H(2,1)(X_2 - X_1) \\ &\vdots \\ Z_t &= \frac{1}{\sigma}\sum_{s=1}^{t-1}\kappa_H(t,s)(X_{s+1} - X_s) \quad \text{for } t = 2, 3, \ldots, n. \end{aligned} \tag{9}$$

Using this discretization, now we can rewrite the MLE and get an approximation to it.

4.2 Approximation to the MLE

Discretization of Q using the alternate form in (7) leads to

$$\begin{aligned} Q_1 &= u_1^{2H-1} Z_1 \\ Q_2 &= u_2^{2H-1} Z_2 + u_1^{2H-1}(Z_2 - Z_1) \\ &\vdots \\ Q_t &= u_t^{2H-1} Z_t + \sum_{s=1}^{t-1} u_s^{2H-1}(Z_{s+1} - Z_s) \quad \text{for } t = 2, 3, \ldots, n. \end{aligned}$$

Hence the maximum likelihood estimator can be approximated to

$$\hat{\theta}_T \approx \hat{\theta}_n = \frac{V_1(n)}{V_2(n)}, \tag{10}$$

where

$$V_1(n) = \sum_{s=1}^{n-1} Q_s(Z_{s+1} - Z_s)$$

and

$$V_2(n) = \sum_{s=2}^{n-1} Q_s^2(w_{s+1}^H - w_s^H).$$

It may be noted that this is an approximation and hence it will not have the optimal properties of the MLE.

4.3 Estimating equation

In this subsection, estimator of θ is obtained by solving the optimal estimating equation. Since we apply the estimating function approach to data which is already discretized, we do not loose anything in terms of the optimality of estimating function. Towards deriving an optimal estimating function, first we obtain a representation for $F_{i-1}(\theta)$ as follows

$$F_{i-1}(\theta) = A_i + B_i + C_i\theta,$$

where

$$\begin{aligned} A_i &= \tfrac{1}{\sigma}\sum_{s=1}^{i-2}\kappa_H(i,s)(X_{s+1}-X_s),\\ B_i &= \tfrac{\kappa_H(i,i-1)}{\sigma}\left\{\tfrac{u_i^{2H}-u_{i-1}^{2H}-(u_i-u_{i-1})^{2H}}{2u_{i-1}^{2H}}\right\}X_{i-1}\\ \text{and}\quad C_i &= \tfrac{\kappa_H(i,i-1)}{\sigma}\left\{X_{i-1}(u_i-u_{i-1})+\left[\tfrac{u_{i-1}^{2H}-u_i^{2H}+(u_i-u_{i-1})^{2H}}{2u_{i-1}^{2H}}\right]\sum_{s=1}^{i-2}X_s(u_{s+1}-u_s)\right\}, \end{aligned} \tag{11}$$

(see (A7) of Appendix a for details).

This leads to,

$$\dot{F}_{i-1}(\theta) = C_i.$$

Now using (5), we obtain estimating function as

$$G_t^*(\theta) = \sum_{i=2}^{t}\frac{C_i}{\phi_{i-1}}\left\{Z_i - (A_i + B_i + C_i\theta)\right\}.$$

For brevity we denote $\phi_{t-1}(\theta)$ by ϕ_{t-1}. Note that

$$A_t = Z_t - D_t,$$

where

$$D_t = \frac{1}{\sigma}\kappa_H(t,t-1)(X_t - X_{t-1}).$$

Therefore,

$$G_t^*(\theta) = \sum_{i=2}^{t}\frac{C_i}{\phi_{i-1}}\left\{D_i - B_i - C_i\theta\right\}.$$

Equating the above estimating function to 0, we can obtain $\hat{\theta}$ as,

$$\hat{\theta} = \frac{\sum_{i=2}^{t}\left(C_i\left[D_i - B_i\right]/\phi_{i-1}\right)}{\sum_{i=2}^{t}\left(C_i^2/\phi_{i-1}\right)}. \tag{12}$$

4.4 Limiting distribution

Define $S_{ni} = s_n^{-1} G_i^*(\theta)$ for $1 \leq i \leq n$, where s_n is the standard deviation of G_n^*. Let $Y_{ni} = S_{ni} - S_{ni-1}$ be the differences of standardized $G^*(\theta)$ and $U_{ni}^2 = \sum_{j=1}^{i} Y_{nj}^2$ be the cumulative sum of difference squares. With the introduction of these, we state the following theorems from Hall and Heyde (1980, pp. 58-64).

Theorem 1. *Let $\{S_{ni}, \mathcal{F}_{ni} : 1 \leq i \leq k_n, n \geq 1\}$ be a zero-mean, square-integrable martingale array with differences Y_{ni}, and let η^2 be an a.s. finite r.v. Suppose that the following conditions hold,*

(i) $\max_i |Y_{ni}| \xrightarrow{p} 0$,

(ii) $\sum_i Y_{ni}^2 \xrightarrow{p} \eta^2$,

(iii) $E\left(\max_i Y_{ni}^2\right)$ is bounded in n, and

(iv) *the σ -fields are nested:* $\mathcal{F}_{ni} \subseteq \mathcal{F}_{n+1,i}$, for $1 \leq i \leq k_n,\ n \geq 1$.

Then,

$$S_{nk_n} = \sum_i Y_{ni} \xrightarrow{d} Z,$$

where the r.v. Z has characterstic function $E\ exp(-\frac{1}{2}\eta^2 t^2)$.

Theorem 2. *Suppose that the conditions of Theorem 1 hold and the* $P\left(\eta^2 > 0\right) = 1$. *Then*

$$S_{nk_n}/U_{nk_n} \xrightarrow{d} N(0,1).$$

The above theorems gives the asymptotic normality of the proposed estimating function.

5 Simulation study

Simulations were carried out to study the performance of the proposed estimating equation. Different total observation times ($T = 1$, 1.5 and 2) and lags ($lag = 0.0005$, 0.0008, 0.0010, 0.0012, 0.0015 and 0.0020) have been considered, with $\theta = 7, \sigma = 1$ and $H = 0.7$. For each combination of T and lag, 50 simulations were considered.

All computations were carried out using R software (version 2.9.2). For simulation of fBm, we have used the fact that fractional Gaussian noise (fGn) is the stationary increment of fBm (cf. Beran, 1994, pp. 55). The fGn was simulated using the method based on fast Fourier transform (FFT) (Beran, 1994, pp. 216). The parameter θ was estimated using the proposed estimating function approach. For each combination of T and lag, the mean, and the mean square errors of the estimator in (12) are reported in Tables 1. Table 2 compares the mean and the mean square error of the proposed estimator and the estimate obtained as an approximation to the MLE in (10). Only few combinations of T and lag were considered as the MLE computations need huge amount of computing time to arrive at the estimator.

TABLE 1 AND 2 HERE

From Table 1 it may be noted that for all chosen combinations, the performance of the proposed estimator improves with the increase in T. Also it can be seen from Table 2 that the performance of the proposed estimator is comparable with the one obtained by approximating MLE. However, the time taken for the computation of the proposed estimators was far less.

6 Applications

In this section we discuss two applications of the above estimation procedure.

6.1 Fractional Gompertzian model

Gompertzian Model was introduced by Benjamin Gompertz (1825) to analyze the population dynamics and to determine the life contingencies. Later, this model was found suitable in explaining various growth phenomena. Ferrante et. al. (2000) have given the MLEs of the parameters involved in a stochastic Gompertzian model. They suugest an approximation to these MLEs using the trapezoidal rule. These estimators are not very much useful in practical situations, where the process is not observed continuously. Ramanathan and Bhat (2003) suggested estimating functions for the estimation of parameters of this model when the observational scheme is discrete. They have also discussed some of the optimal properties of the suggested estimates.

A fractional version of the stochastic Gompertzian model can be given by the stochastic differential equation

$$dX_t = \{\alpha X_t - \beta X_t \log X_t\}dt + \sigma X_t dW_t^H.$$

This can be alternately expressed as

$$dX_t/X_t = \{\alpha - \beta \log X_t\}dt + \sigma dW_t^H,$$

i.e.,

$$d \log X_t = (\alpha - \beta \log X_t)\, dt + \sigma dW_t^H$$

and hence

$$\log X_t = \int_0^t (\alpha - \beta \log X_t)\, dt + \sigma W_t^H,$$

which is nothing but the fractional Ornstein-Uhlenbeck type process. The above SDE is similar to (8). Hence the stimating equation can be obtained in a similar fashion. The details of this derivation is given in Appendix B.

6.2 Fractional generalization of Vasicek model

Vasicek model is used in finance for modeling interest rates. Interest rates generally tend to be highly positively autocorrelated with long swings having sample autocorrelations that die out slowly. For modeling such a phenomena, it is worth considering the fractional analogue of the Vasicek model (cf. Hog and Frederikson (2006) and Jaworska (2008)). Suppose that instantaneous interest rate Y_t satisfy a stochastic differential equation governed by a fBm W^H.

$$dY_t = \{\theta_1 + \theta_2 Y_t\}dt + \sigma dW_t^H.$$

Here we have two unknown parameters, $\boldsymbol{\theta} = \{\theta_1, \theta_2\}$, which necessitates an extension of the derivation of estimating functions to the multi-parameter case. Towards this, we consider the following estimating function which is a zero-mean martingale,

$$\tilde{G_n^k}(\boldsymbol{\theta}) = \sum_{i=2}^{n} \dot{Q}_{i-1}^k(\boldsymbol{\theta}, H)\big[Z_i - F(Z_{i-1}; \boldsymbol{\theta})\big], \quad \text{k=1,2},$$

where $\dot{Q}^k_{i-1}(\boldsymbol{\theta}, H)$ denotes the derivative of $Q_{i-1}(\boldsymbol{\theta}, H)$ with respect to θ_k. Now consider the class of estimating functions $\mathcal{G}_2$ such that $\mathbf{G}_n = \big(G^1_n(\boldsymbol{\theta}), G^2_n(\boldsymbol{\theta})\big) \in \mathcal{G}_2$ if

$$G^k_n(\boldsymbol{\theta}) = \sum_{i=2}^{n} g^k_{i-1}(\boldsymbol{\theta})\big[Z_i - F(Z_{i-1}; \boldsymbol{\theta})\big], \quad \text{k=1,2,}$$

where g_{i-1} is $\mathcal{F}$-measurable and a continuously differentiable function of $\boldsymbol{\theta}$, $i = 1, \ldots, n$. The optimal estimating function, $\mathbf{G}^*_n \in \mathcal{G}_2$ is the one for which

$$(E\dot{\mathbf{G}}_n)^{-1}(E\mathbf{G}_n\mathbf{G}'_n)((E\dot{\mathbf{G}}_n)^{-1})' - (E\dot{\mathbf{G}}^*_n)^{-1}(E\mathbf{G}^*_n\mathbf{G}^{*'}_n)((E\dot{\mathbf{G}}^*_n)^{-1})'$$

is non-negative definite for all $\mathbf{G}_n \in \mathcal{G}_2$. Here $\dot{\mathbf{G}}_n$ is the matrix of first order derivatives of $\mathbf{G}_n$.

7 Concluding Remarks

The problem of estimating the parameters of a process governed by fractional Brownian motion is addressed when the observational scheme is discrete, using the method of estimating functions. Although our discussion assumes the knowledge of the Hurst parameter H, the method can be very well extended to include this as a part of the parameter space and thus the procedure can be extended for the estimation of H. Another possible direction of future investigation could be that of a situation where the process $\{X_t\}$ is governed by a fractional process other than the Brownian motion. This include fractional Levy process and the related stochastic differential equations. Currently we are working on this problem and the results will be reported elsewhere.

Appendix A

Let $\mathbf{W}_t = (W_1^H, W_2^H, \ldots, W_t^H)$. Note that $\boldsymbol{\sigma}(W_1^H, \ldots, W_t^H) = \boldsymbol{\sigma}(X_1^H, \ldots, X_t^H)$, which is same as $\mathcal{F}_t$, defined in Section 3.2.

Using the conditional distribution of Gaussian random variables, we can write

$$
\begin{aligned}
E[W_t^H|\mathcal{F}_{t-1}] &= E[W_t^H] + Cov(W_t^H, \mathbf{W}_{t-1})[Var(\mathbf{W}_{t-1})]^{-1}\left\{\mathbf{W'}_{t-1} - E[\mathbf{W'}_{t-1}]\right\} \\
&= \boldsymbol{\sigma}_t \boldsymbol{\Sigma}_{t-1}^{-1} \mathbf{W'}_{t-1} \\
&= \frac{1}{\sigma}\boldsymbol{\sigma}_t \boldsymbol{\Sigma}_{t-1}^{-1}(\mathbf{X'}_{t-1} - \theta \mathbf{S'}_{t-1}) \\
&= \frac{1}{\sigma}\boldsymbol{\sigma}_t \boldsymbol{\Sigma}_{t-1}^{-1}\mathbf{X'}_{t-1} - \frac{1}{\sigma}\theta\boldsymbol{\sigma}_t \boldsymbol{\Sigma}_{t-1}^{-1}\mathbf{S'}_{t-1}, \quad \text{using (8)} \qquad \text{(A1)}
\end{aligned}
$$

where $\boldsymbol{\sigma}_t$ is $1 \times (t-1)$ vector of covariances, i.e.

$$\boldsymbol{\sigma}_t = \Big(Cov(W_t^H, W_1^H), \ldots\ldots, Cov(W_t^H, W_{t-1}^H)\Big),$$

$\boldsymbol{\Sigma}_{t-1}$ denotes the $(t-1) \times (t-1)$ variance-covariance matrix of $\mathbf{W}_{t-1}^H$, and $\mathbf{S}_{t-1} = (S_1, \ldots, S_{t-1})$ with $S_l = \sum_{s=1}^{l-1} X_s(u_{s+1} - u_s)$.

Similarly

$$
\begin{aligned}
Var(W_t^H|\mathcal{F}_{t-1}) &= Var(W_t^H) - Cov(W_t^H, \mathbf{W}_{t-1})[Var(\mathbf{W}_{t-1})]^{-1}[Cov(W_t^H, \mathbf{W}_{t-1})]' \\
&= u_t^{2H} - \boldsymbol{\sigma}_t \boldsymbol{\Sigma}_{t-1}^{-1}\boldsymbol{\sigma}'_t. \qquad \text{(A2)}
\end{aligned}
$$

The computation of (A1) and (A2) involves the computation of the inverse of the variance-covariance matrix $\boldsymbol{\Sigma}_{t-1}$, which is difficult in view of its high dimension, therefore we approximate the above expressions by conditioning W_t only on W_{t-1}. One should also note that in any case the entire past comes in through X_t's. Thus,

$$
\begin{aligned}
E[W_t^H|W_{t-1}^H] &= \frac{1}{2}\left\{\frac{u_t^{2H} + u_{t-1}^{2H} - (u_t - u_{t-1})^{2H}}{u_{t-1}^{2H}}\right\} W_{t-1}^H \\
&= \left\{\frac{u_t^{2H} + u_{t-1}^{2H} - (u_t - u_{t-1})^{2H}}{2\sigma u_{t-1}^{2H}}\right\}\left[X_{t-1} - \theta\sum_{s=1}^{t-2} X_s(u_{s+1} - u_s)\right]
\end{aligned}
$$

and

$$Var(W_t^H|W_{t-1}^H) = u_t^{2H} - \frac{\left[u_t^{2H} + u_{t-1}^{2H} - (u_t - u_{t-1})^{2H}\right]^2}{4u_{t-1}^{2H}}.$$

From (8)

$$E[X_t|\mathcal{F}_{t-1}] = \theta \sum_{s=1}^{t-1} X_s(u_{s+1} - u_s) + \sigma E[W_t^H|W_{t-1}^H] \tag{A3}$$

and

$$E[X_t^2|\mathcal{F}_{t-1}] = \theta^2 \left[\sum_{s=1}^{t-1} X_s(u_{s+1} - u_s)\right]^2 + \sigma^2 E[W_t^{H^2}|W_{t-1}^H] + 2\sigma\theta E[W_t^H|W_{t-1}^H] \sum_{s=1}^{t-1} X_s(u_{s+1} - u_s). \tag{A4}$$

Now from (9)

$$Z_t = \frac{1}{\sigma} \sum_{s=1}^{t-1} \kappa_H(t,s)(X_{s+1} - X_s) \quad \text{for } t = 2, 3, \ldots, n$$

or

$$\sigma Z_t = \left[\sum_{s=1}^{t-2} \kappa_H(t,s)(X_{s+1} - X_s) - \kappa_H(t,t-1)X_{t-1}\right] + \kappa_H(t,t-1)X_t. \tag{A5}$$

Taking expectation on both sides,

$$\sigma E\big[Z_t|\mathcal{F}_{t-1}\big] = \left[\sum_{s=1}^{t-2} \kappa_H(t,s)(X_{s+1} - X_s) - \kappa_H(t,t-1)X_{t-1}\right] + \kappa_H(t,t-1)E[X_t|\mathcal{F}_{t-1}] \tag{A6}$$

$$= \sum_{s=1}^{t-2} \kappa_H(t,s)(X_{s+1} - X_s) + \kappa_H(t,t-1)\left\{E[X_t|\mathcal{F}_{t-1}] - X_{t-1}\right\}$$

$$= \sum_{s=1}^{t-2} \kappa_H(t,s)(X_{s+1} - X_s) + \kappa_H(t,t-1)\left[\theta \sum_{s=1}^{t-1} X_s(u_{s+1} - u_s) + \sigma E[W_t^H|W_{t-1}^H] - X_{t-1}\right]$$

$$= \sum_{s=1}^{t-2} \kappa_H(t,s)(X_{s+1} - X_s) + \kappa_H(t,t-1)\left[\theta \sum_{s=1}^{t-1} X_s(u_{s+1} - u_s) + \left\{\frac{u_t^{2H} + u_{t-1}^{2H} - (u_t - u_{t-1})^{2H}}{2u_{t-1}^{2H}}\right\}\left(X_{t-1} - \theta \sum_{s=1}^{t-2} X_s(u_{s+1} - u_s)\right) - X_{t-1}\right]$$

$$= \sum_{s=1}^{t-2} \kappa_H(t,s)(X_{s+1} - X_s) + \kappa_H(t,t-1) \Bigg[X_{t-1} \left(\frac{u_t^{2H} + u_{t-1}^{2H} - (u_t - u_{t-1})^{2H}}{2u_{t-1}^{2H}} - 1 \right) + \theta \left(\sum_{s=1}^{t-1} X_s(u_{s+1} - u_s) - \left[\frac{u_t^{2H} + u_{t-1}^{2H} - (u_t - u_{t-1})^{2H}}{2u_{t-1}^{2H}} \right] \sum_{s=1}^{t-2} X_s(u_{s+1} - u_s) \right) \Bigg]$$

$$= \sum_{s=1}^{t-2} \kappa_H(t,s)(X_{s+1} - X_s) + \kappa_H(t,t-1) \Bigg[\left\{ \frac{u_t^{2H} - u_{t-1}^{2H} - (u_t - u_{t-1})^{2H}}{2u_{t-1}^{2H}} \right\} X_{t-1} + \theta \left\{ X_{t-1}(u_t - u_{t-1}) + \left[1 - \frac{u_t^{2H} + u_{t-1}^{2H} - (u_t - u_{t-1})^{2H}}{2u_{t-1}^{2H}} \right] \sum_{s=1}^{t-2} X_s(u_{s+1} - u_s) \right\} \Bigg]$$

$$= \sum_{s=1}^{t-2} \kappa_H(t,s)(X_{s+1} - X_s) + \kappa_H(t,t-1) \Bigg[\left\{ \frac{u_t^{2H} - u_{t-1}^{2H} - (u_t - u_{t-1})^{2H}}{2u_{t-1}^{2H}} \right\} X_{t-1} + \theta \left\{ X_{t-1}(u_t - u_{t-1}) + \left[\frac{u_{t-1}^{2H} - u_t^{2H} + (u_t - u_{t-1})^{2H}}{2u_{t-1}^{2H}} \right] \sum_{s=1}^{t-2} X_s(u_{s+1} - u_s) \right\} \Bigg]$$

Therefore,

$$F_{t-1}(\theta) = E\big[Z_t | \mathcal{F}_{t-1}\big] = A_t + B_t + C_t\theta$$

and $$\dot{F}_{t-1}(\theta) = C_t,$$

where

$$A_t = \frac{1}{\sigma} \sum_{s=1}^{t-2} \kappa_H(t,s)(X_{s+1} - X_s),$$
$$B_t = \frac{\kappa_H(t,t-1)}{\sigma} \left\{ \frac{u_t^{2H} - u_{t-1}^{2H} - (u_t - u_{t-1})^{2H}}{2u_{t-1}^{2H}} \right\} X_{t-1}$$

and $$C_t = \frac{\kappa_H(t,t-1)}{\sigma} \left\{ X_{t-1}(u_t - u_{t-1}) + \left[\frac{u_{t-1}^{2H} - u_t^{2H} + (u_t - u_{t-1})^{2H}}{2u_{t-1}^{2H}} \right] \sum_{s=1}^{t-2} X_s(u_{s+1} - u_s) \right\}. \tag{A7}$$

Now again using (A5),

$$\sigma^2 E\big[Z_t^2 | \mathcal{F}_{t-1}\big] = \left[\sum_{s=1}^{t-2} \kappa_H(t,s)(X_{s+1} - X_s) - \kappa_H(t,t-1)X_{t-1} \right]^2 + \kappa_H(t,t-1)^2 E[X_t^2 | \mathcal{F}_{t-1}] + 2\kappa_H(t,t-1)E[X_t | \mathcal{F}_{t-1}] \left[\sum_{s=1}^{t-2} \kappa_H(t,s)(X_{s+1} - X_s) - \kappa_H(t,t-1)X_{t-1} \right]. \tag{A8}$$

By definition,

$$\begin{aligned}\phi_{t-1}(\theta) &= E_\theta\left[\left(Z_t - F_{t-1}(\theta)\right)^2 | \mathcal{F}_{t-1}\right] \\ &= E_\theta\left[\left(Z_t - E_\theta\left[Z_t|\mathcal{F}_{t-1}\right]\right)^2 |\mathcal{F}_{t-1}\right] \\ &= E\left[Z_t^2|\mathcal{F}_{t-1}\right] - \left(E\left[Z_t|\mathcal{F}_{t-1}\right]\right)^2.\end{aligned}$$

Therefore, from (A6) and (A8)

$$\begin{aligned}\sigma^2\phi_{t-1}(\theta) &= \kappa_H(t,t-1)^2\left[E[X_t^2|\mathcal{F}_{t-1}] - (E[X_t|\mathcal{F}_{t-1}])^2\right] \\ &= \kappa_H(t,t-1)^2\sigma^2\left[E[{W_t^H}^2|W_{t-1}^H] - \left(E[W_t^H|W_{t-1}^H]\right)^2\right], \\ &\quad \text{(using (A3) and (A4)).}\end{aligned}$$

Hence,

$$\begin{aligned}\phi_{t-1}(\theta) &= \kappa_H(t,t-1)^2 Var(W_t^H|W_{t-1}^H) \\ &= \kappa_H(t,t-1)^2\left[u_t^{2H} - \frac{\left[u_t^{2H} + u_{t-1}^{2H} - (u_t - u_{t-1})^{2H}\right]^2}{4u_{t-1}^{2H}}\right]\end{aligned}$$

Appendix B

Let $\begin{pmatrix} X \\ Y \end{pmatrix} \sim BVN(0, 0, \sigma_X^2, \sigma_Y^2, \rho)$.

Then

$$E\left[e^Y \middle| X\right] = \exp[x\rho\sigma_Y/\sigma_X] \exp\left[-\tfrac{1}{2}(1-\rho^2)\sigma_Y^2\right]$$

and

$$V\left[e^Y \middle| X\right] = f(f-1)\exp[2x\rho\sigma_Y/\sigma_X],$$

where $f = \exp\left[(1-\rho^2)\sigma_Y^2\right]$ (cf. Balakrishnan and Lai (2009)).

Therefore for the fBm,

$$E\left[e^{\sigma W_t^H} \middle| W_{t-1}^H\right] = \exp\left\{W_{t-1}^H \rho\sigma[t/(t-1)]^H\right\} \exp\left\{-\tfrac{1}{2}(1-\rho^2)\sigma^2 t^{2H}\right\}$$

and

$$V\left[e^{\sigma W_t^H} \middle| W_{t-1}^H\right] = f(f-1)\exp\left\{2W_{t-1}^H \rho\sigma[t/(t-1)]^H\right\}.$$

Note that, here $\rho = \frac{t^{2H}+(t-1)^{2H}-1}{[t(t-1)]^H}$,

(here again we have conditioned only on W_{t-1}^H).

We discretize the $\log X_t$ and hence the X_t process as follows.

Let

$$\begin{aligned}
\log X_1 &= 0 \\
\log X_2 &= (\alpha - \beta \log X_1)(u_2 - u_1) + \sigma W_2^H \\
&\vdots
\end{aligned}$$

and

$$\log X_t = \sum_{s=1}^{t-1} (\alpha - \beta \log X_s)(u_{s+1} - u_s) + \sigma W_t^H.$$

Therefore

$$X_t = \exp\left[\sum_{s=1}^{t-1} (\alpha - \beta \log X_s)(u_{s+1} - u_s) + \sigma W_t^H\right]$$

$$= \exp\left[\sum_{s=1}^{t-1} (\alpha - \beta \log X_s)(u_{s+1} - u_s)\right] . e^{\sigma W_t^H}$$

Now we compute the following terms which are needed for the estimating equation.

$$E[X_t|\mathcal{F}_{t-1}] = \exp\left[\sum_{s=1}^{t-1} (\alpha - \beta \log X_s)(u_{s+1} - u_s)\right] . \, E\left[e^{\sigma W_t^H} \middle| W_{t-1}^H\right],$$

and

$$E[X_t^2|\mathcal{F}_{t-1}] = \exp\left[2\sum_{s=1}^{t-1}(\alpha-\beta\log X_s)(u_{s+1}-u_s)\right].\ E\left[\left(e^{\sigma W_t^H}\right)^2\middle| W_{t-1}^H\right].$$

Therefore,

$$\begin{aligned}
&E[X_t^2|\mathcal{F}_{t-1}] - (E[X_t|\mathcal{F}_{t-1}])^2 = \\
&\quad \exp\left[2\sum_{s=1}^{t-1}(\alpha-\beta\log X_s)(u_{s+1}-u_s)\right]\left\{E\left[\left(e^{\sigma W_t^H}\right)^2\middle| W_{t-1}^H\right] - \left(E\left[e^{\sigma W_t^H}\middle| W_{t-1}^H\right]\right)^2\right\} \\
&= \quad \exp\left[2\sum_{s=1}^{t-1}(\alpha-\beta\log X_s)(u_{s+1}-u_s)\right] V\left[e^{\sigma W_t^H}\middle| W_{t-1}^H\right].
\end{aligned}$$

Now, for the $\{Z_t\}$ process,

$$\begin{aligned}
\sigma E\big[Z_t|\mathcal{F}_{t-1}\big] &= \left[\sum_{s=1}^{t-2}\kappa_H(t,s)(X_{s+1}-X_s) - \kappa_H(t,t-1)X_{t-1}\right] + \kappa_H(t,t-1)E[X_t|\mathcal{F}_{t-1}] \\
&= \left[\sum_{s=1}^{t-2}\kappa_H(t,s)(X_{s+1}-X_s) - \kappa_H(t,t-1)X_{t-1}\right] \\
&\qquad + \kappa_H(t,t-1)\exp\left[\sum_{s=1}^{t-1}(\alpha-\beta\log X_s)(u_{s+1}-u_s)\right].\ E\left[e^{\sigma W_t^H}\middle| W_{t-1}^H\right] \\
&= \left[\sum_{s=1}^{t-2}\kappa_H(t,s)(X_{s+1}-X_s) - \kappa_H(t,t-1)X_{t-1}\right] \\
&\qquad + \kappa_H(t,t-1)\exp\left[\sum_{s=1}^{t-1}(\alpha-\beta\log X_s)(u_{s+1}-u_s)\right]. \\
&\qquad\qquad \exp\left[W_{t-1}^H\rho\sigma(t/t-1)^H\right]\exp\left[-\frac{1}{2}(1-\rho^2)\sigma^2 t^{2H}\right] \\
&= \left[\sum_{s=1}^{t-2}\kappa_H(t,s)(X_{s+1}-X_s) - \kappa_H(t,t-1)X_{t-1}\right] \\
&\qquad + \kappa_H(t,t-1)\exp\left[\sum_{s=1}^{t-1}(\alpha-\beta\log X_s)(u_{s+1}-u_s)\right. \\
&\qquad\qquad \left. + W_{t-1}^H\rho\sigma(t/t-1)^H - \frac{1}{2}(1-\rho^2)\sigma^2 t^{2H}\right].
\end{aligned}$$

Also,

$$\begin{aligned}
\dot{F}_{t-1}(\alpha) = \tfrac{1}{\sigma}\kappa_H(t,t-1)(u_t-u_1)\exp&\left[\sum_{s=1}^{t-1}(\alpha-\beta\log X_s)(u_{s+1}-u_s)\right. \\
&\left. + W_{t-1}^H\rho\sigma(t/t-1)^H - \tfrac{1}{2}(1-\rho^2)\sigma^2 t^{2H}\right],
\end{aligned}$$

and

$$\sigma^2\phi_{t-1}(\alpha) = \kappa_H(t,t-1)^2\left[E[X_t^2|\mathcal{F}_{t-1}] - (E[X_t|\mathcal{F}_{t-1}])^2\right]$$
$$= \kappa_H(t,t-1)^2 \exp\left[2\sum_{s=1}^{t-1}(\alpha-\beta\log X_s)(u_{s+1}-u_s)\right] V\left[e^{\sigma W_t^H}\,\middle|\, W_{t-1}^H\right]$$
$$= \kappa_H(t,t-1)^2 \exp\left[2\sum_{s=1}^{t-1}(\alpha-\beta\log X_s)(u_{s+1}-u_s)\right].$$
$$f(f-1)\exp\left[2W_{t-1}^H\rho\sigma(t/t-1)^H\right]$$
$$= \kappa_H(t,t-1)^2 f(f-1)\exp\left[2\sum_{s=1}^{t-1}(\alpha-\beta\log X_s)(u_{s+1}-u_s) + 2W_{t-1}^H\rho\sigma(t/t-1)^H\right].$$

Therefore

$$\frac{\dot{F}_{t-1}(\alpha)}{\phi_{t-1}(\alpha)} = \frac{M_t}{N_t}, \tag{B1}$$

where

$$M_t = \sigma(u_t - u_1),$$

and

$$N_t = \kappa_H(t,t-1)f(f-1)\exp\left[\sum_{s=1}^{t-1}(\alpha-\beta\log X_s)(u_{s+1}-u_s) + W_{t-1}^H\rho\sigma(t/t-1)^H + \frac{1}{2}(1-\rho^2)\sigma^2 t^{2H}\right].$$

Also,

$$Z_t - E\big[Z_t|\mathcal{F}_{t-1}\big] = \frac{\kappa_H(t,t-1)}{\sigma}\left\{X_t - \exp\left[\sum_{s=1}^{t-1}(\alpha-\beta\log X_s)(u_{s+1}-u_s) + W_{t-1}^H\rho\sigma(t/t-1)^H - \tfrac{1}{2}(1-\rho^2)\sigma^2 t^{2H}\right]\right\}. \tag{B2}$$

Using (B1) and (B2) we get the required estimating equation, as in (5).

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Table 1: Estimates from Estimating functions and their MSE

T	lag	$time\ pts$	$mean$	MSE
1	0.0005	2000	7.01657	0.018267610
1.5	0.0005	3000	6.99959	0.000008689
2	0.0005	4000	7.00001	0.000000019
1	0.0008	1250	7.00389	0.003729347
1.5	0.0008	1875	7.00002	0.000010367
2	0.0008	2500	6.99997	0.000000014
1	0.0010	1000	6.96557	0.083797990
1.5	0.0010	1500	6.99961	0.000003353
2	0.0010	2000	7.00001	0.000000002
1	0.0012	833	6.88753	0.242017300
1.5	0.0012	1250	6.99869	0.000059507
2	0.0012	1666	7.00000	0.000000004
1	0.0015	666	6.99188	0.002379316
1.5	0.0015	1000	6.99985	0.000003880
2	0.0015	1333	7.00000	0.000000002
1	0.0020	500	6.87364	0.899578800
1.5	0.0020	750	7.00023	0.000003128
2	0.0020	1000	6.99999	0.000000005

Table 2: Comparisons of the proposed estimator and the MLE in eq.(10)

			$Proposed\ estimator$		MLE	
T	lag	$time\ pts$	$mean$	MSE	$mean$	MSE
1	0.0010	1000	6.96557	0.083797990	7.00784	0.01833146
1.5	0.0010	1500	6.99961	0.000003353	7.00065	0.000006624
2	0.0010	2000	7.00001	0.000000002	7.00066	0.000000445