Minimum Variance Unbiased Estimation in the Rayleigh Distribution under Progressive Type II Censored Data with Binomial Removals.

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Abstract

This paper concerns with the problem of uniformly minimum variance unbiased estimation of the scale parameter of Rayleigh distribution based on progressive Type II censored data with binomial removals. We obtain the uniformly minimum variance unbiased estimator (UMVUE) for powers of the scale parameter and its functions. The UMVUE of the variance of these estimators are also given. The UMVUE of the (i) mode (ii) r^{th} moment (iii) mean (iv) variance (v) hazard function (vi) median (vii) p^{th} quantile (viii) p.d.f. (ix) reliability function and (x) c.d.f. of the Rayleigh distribution are derived. The UMVUE of p.d.f. is utilized to obtain the UMVUE of $P(X < Y)$. An illustrative numerical example is presented.

Keywords: progressive Type II censored sample, Rayleigh distribution, binomial distribution, complete sufficient statistic, UMVUE.

Mathematics Subject Classification: 62N01, 62N02.

1 Introduction

A Type II censored sample is one for which only m smallest observations in a sample of n items are observed. A generalization of Type II censoring is a progressive Type II censoring. Under this scheme, n units of the same kind are placed on test at time zero, and m failures are observed. When the first failure is observed, a number of r_1 surviving units are randomly withdrawn from the test; at the second failure time, r_2 surviving units are selected at random and taken out of the experiment, and so on. At the time of m^{th} failure, the remaining $r_m = n - r_1 - r_2 - \cdots - r_{m-1} - m$ units are removed. Balakrishnan *et.al* [5] indicated that such scheme can arise in clinical trials where the drop out of patients may be caused by migration or by lack of interest. In such situations, the progressive censoring scheme with random removals is required. For a detailed discussion of progressive censoring we refer to Balakrishnan and Aggarwala [4] and Balakrishnan [2]. If $r_1 = r_2 = \cdots = r_{m-1} = 0$, then, this scheme reduces to the Type II censoring scheme. Also note that if $r_1 = r_2 = \cdots = r_m = 0$,so that $m = n$,this scheme reduces to the case of no censoring that is the case of a complete sample. In this paper, we use progressive Type II censoring scheme with binomial removals where the number of units removed at each failure time follows a binomial distribution.

The Rayleigh distribution was first derived by Lord Rayleigh in connection with a study of acoustical problems. Since then many investigators have used the Rayleigh distribution or some related forms of it in a variety of engineering, wave propagation, radiation and analysis of target data studies. The Rayleigh distribution is also used to model wave heights in oceanography, and in communication theory to describe hourly median and instantaneous peak power of received radio signals. Several such situations have been discussed by Polovko [20] , Longuet-Higgins [19], Zhi Ren *et.al* [22] , Uvaison and Grodzenskaya [26], Yinhui Deng *et.al* [8], Eryi Hu [10], Takeshi Yamane [27], and many others. The Rayleigh distribution is a special case of two parameter Weibull distribution. The hazard rate of this distribution is linearly increasing function of time. For a review of literature on estimating parameters of the Rayleigh distribution one may refer to Lee *et.al* [18], Dyer [9], Lalitha and Mishra [15], Kim and Han [14], Balakrishnan N. [3], Solimman [24], Raqab and Madi [21], and many others.

Inference for Rayleigh distribution based on progressive Type II censored data were discussed by many authors. Ali Mousa *et.al* [1] obtained the maximum likelihood and Bayes estimates for one and two parameters and the reliability function of Rayleigh distribution under progressive Type II censored samples. Kim *et.al* [13] have obtained the maximum likelihood estimator, Bayes estimator and credible intervals for the scale parameter and reliability function of the Rayleigh distribution based on general progressive Type II censored data. Lee [16] have obtained the UMVUE of lifetime performance index based on the type II multiply censored sample and develop the hypothesis testing procedure. Lee *et.al* [17] have obtained MLE of lifetime performance index and developed Baysian test for it based on progressive Type II right censored sample.

In this paper our objective is to study the UMVU estimation of the scale parameter and its various functions of Rayleigh distribution based on progressive Type II censored data with binomial removals. This paper is organized as follows. In Section 2, the likelihood function is given. In Section 3, the UMVUE of parameter of θ and its functions are derived . Also, the UMVUE of the (i) mode (ii) r^{th} moment (iii) mean and variance (iv) hazard function (v) median (vi) p^{th} quintile (vii) positive power of reliability function (viii) p.d.f. (ix) c.d.f. are obtained. In Section 4 using the UMVUE of p.d.f. the UMVUE of $P(X<Y)$ is derived. In Section 5, an illustrative numerical example is given.

2 The model

Let the failure time distribution be Rayleigh with probability density function,

$$f(x)=\begin{cases}\dfrac{x}{\theta^2}\exp\left[\dfrac{-x^2}{2\theta^2}\right] & 0<x<\infty\ ,\ \theta>0\\ 0 & \text{otherwise}\end{cases} \tag{1}$$

where θ is a scale parameter. The density (1) is given in [12].

The cumulative distribution function is given by,

$$F(x)=1-\exp\left[\frac{-x^2}{2\theta^2}\right],\quad 0<x<\infty \tag{2}$$

The survival function is given by,

$$S(x)=\exp\left[\frac{-x^2}{2\theta^2}\right],\quad 0<x<\infty \tag{3}$$

The density given in (1) can be written as,

$$f(x)=\frac{a(x)\left[h(\theta)\right]^{d(x)}}{\left[g(\theta)\right]} \tag{4}$$

where $a(x)=x,\quad h(\theta)=\exp\left[\dfrac{-1}{2\theta^2}\right],\quad d(x)=x^2$ and $g(\theta)=\theta^2$ (5)

such that $a(x)>0$ and $g(\theta)=\int_0^\infty a(x)\left[h(\theta)\right]^{d(x)}dx$.

Let $(X_1, R_1), (X_2, R_2), \cdots, (X_m, R_m)$,denote a progressively Type II censored sample, where $X_i = X_{i:m:n}$, for $i = 1, 2, \cdots m.$ and $X_1 < X_2 < \cdots < X_m$. The conditional likelihood function can be written as, see Cohen[7],

$$L(\theta; \underline{x} / \underline{R} = \underline{r}) = c \prod_{i=1}^{m} f(x_i) \cdot [s(x_i)]^{r_i} \tag{6}$$

where $c = n(n - r_1 - 1)(n - r_1 - r_2 - 2) \cdots (n - r_1 - r_2 - \cdots r_{m-1} - m + 1)$,

and $0 \le r_i \le (n - m - r_1 - r_2 - \cdots r_{i-1})$ for $i = 1, 2, \cdots m - 1.$

Substituting (1) and (3) in (6) we get,

$$L(\theta; \underline{x} / \underline{R} = \underline{r}) = \frac{c}{\theta^{2m}} \left(\prod_{i=1}^{m} x_i \right) \exp\left\{ -\frac{1}{2\theta^2} \sum_{i=1}^{m} (1 + r_i) x_i^2 \right\} \tag{7}$$

We assume that X_i and R_i are independent for all i .We further suppose that the number of units removed at each failure time follows a binomial distribution with probability *p*. From Tse *et al.* [28] the joint probability mass function of $r_1, r_2, \ldots r_{m-1}$ is given by,

$$P(\underline{R} = \underline{r}) = \frac{(n-m)!\, p^{\sum_{i=1}^{m-1} r_i} (1-p)^{(m-1)(n-m) - \sum_{i=1}^{m-1} (m-i) r_i}}{\left(n - m - \sum_{i=1}^{m-1} r_i \right)! \prod_{i=1}^{m-1} r_i !} \tag{8}$$

That is

$$P(\underline{R} = \underline{r}) = \frac{(n-m)! \left[\dfrac{p}{(1-p)^m} \right]^{\sum_{i=1}^{m-1} r_i} (1-p)^{\sum_{i=1}^{m-1} i r_i}}{\prod_{i=1}^{m} r_i ! \left[(1-p)^{-(n-m)} \right]^{(m-1)}} \tag{9}$$

The unconditional likelihood function is,

$$L(\theta, p; \underline{x}, \underline{r}) = L(\theta; \underline{x} / \underline{R} = \underline{r}) P(\underline{R} = \underline{r}) \tag{10}$$

Using (7) and (9) in (10) we can write the full likelihood function as,

$$L(\theta, p; \underline{x}, \underline{r}) = \frac{c}{\theta^{2m}} \left(\prod_{i=1}^{m} x_i \right) \exp\left\{ -\frac{1}{2\theta^2} \sum_{i=1}^{m} (1+r_i) x_i^2 \right\}$$

$$\times \frac{(n-m)! \left[\frac{p}{(1-p)^m} \right]^{\sum_{i=1}^{m-1} r_i} (1-p)^{\sum_{i=1}^{m-1} i r_i}}{\prod_{i=1}^{m} r_i! \left[(1-p)^{-(n-m)} \right]^{(m-1)}}$$

3 Unbiased estimation

Let $Y_i = X_i^2, i = 1, 2, \ldots, m$ (11)

then Y_i have exponential distribution with mean $2\theta^2$ .Now consider the following transformation,

$$Z_1 = nY_1$$
$$Z_i = (n - i + 1 - r_1 - r_2 - \cdots - r_{i-1})(Y_i - Y_{i-1}) \;, i = 2, 3, \ldots, m. \tag{12}$$

In order to derive the distribution of Z_i, $i = 1, 2, \cdots, m$ consider the inverse transformation $Y_1 = \frac{Z_1}{n}$ and $Y_i = \sum_{i=2}^{m} \frac{Z_i}{(n - r_{i-1} - i + 1)}$, $i = 2, 3, \cdots m$. The variables $Z_1, Z_2, \cdots Z_m$ defined in (12) are all independent and identically distributed with exponential distribution with mean $2\theta^2$, see [25]. The joint density of $Z_1, Z_2, \cdots, Z_m$ is,

$$f(\underline{z}, \theta / \underline{R} = \underline{r}) = \left(\frac{1}{2\theta^2} \right)^m \exp\left\{ -\frac{1}{2\theta^2} \sum_{i=1}^{m} z_i \right\} \tag{13}$$

It can be seen that,

$$\sum_{i=1}^{m} z_i = \sum_{i=1}^{m} (1 + r_i) y_i \tag{14}$$

Using (11) in (14) we have $\sum_{i=1}^{m} z_i = \sum_{i=1}^{m} (1 + r_i) x_i^2$ (15)

Let $T = \sum_{i=1}^{m} Z_i$

$$= \sum_{i=1}^{m} (1 + r_i) x_i^2 \tag{16}$$

Since (13) is a member of exponential family of distributions, T is a complete sufficient statistic for θ. The distribution of T is gamma with parameters $\frac{1}{2\theta^2}$ and m, which is again a member of exponential family of distributions. The p.d.f. of T is given by,

$$f(t,\theta) = \frac{B(t,m)\left[h(\theta)\right]^t}{\left[g(\theta)\right]^m} \tag{17}$$

where $B(t,m) = \frac{t^{m-1}}{2^m \Gamma m}$, $h(\theta) = \exp\left[\frac{-1}{2\theta^2}\right]$, $g(\theta) = \theta^2$.

Jani and Dave [11] have studied the problem of minimum variance unbiased estimation in a class of exponential family of distributions. They have shown that if $X_1, X_2, \cdots X_n$ be a random sample from density of the type given in (4) and the p.d.f. of its complete sufficient statistics can be written as the one given in (17) then the UMVUE of $\left[h(\theta)\right]^k$ is given by,

$$H_{k,n} = \frac{B(t-k,n)}{B(t,n)}, \qquad t > k \tag{18}$$

and the UMVUE of $\left[g(\theta)\right]^k$ is,

$$G_{k,n} = \frac{B(t,n+k)}{B(t,n)} \tag{19}$$

Following the results derived in Jani and Dave [11], we get the UMVUE of some important parametric functions as given below.

(i) Using (18) the UMVUE of $\exp\left[\frac{-1}{2\theta^2}k\right]$ is,

$$H_{K,m} = \left[1 - \frac{k}{\sum_{i=1}^{m}(1+r_i)x_i^2}\right]^{m-1}, \quad \sum_{i=1}^{m}(1+r_i)x_i^2 > k \tag{20}$$

Special case 1: Substituting $k = -2$ in (20) we get the UMVUE of $\exp\left[\frac{1}{\theta^2}\right]$ as,

$$H_{-2,m} = \left[1 + \frac{2}{\sum_{i=1}^{m}(1+r_i)x_i^2}\right]^{m-1} \quad , \quad \sum_{i=1}^{m}(1+r_i)x_i^2 > 0 \tag{21}$$

Special case 2 : Substituting $k = 2$ in (20) we get UMVUE of $\exp\left[\frac{-1}{\theta^2}\right]$ as,

$$H_{2,m} = \left[1 - \frac{2}{\sum_{i=1}^{m}(1+r_i)x_i^2}\right]^{m-1} \quad , \quad \sum_{i=1}^{m}(1+r_i)x_i^2 > 2 \tag{22}$$

(ii)Using (20) the UMVUE of the variance of $H_{K,m}$ is given by,

$$\tilde{V}ar[H_{K,m}] = \left[1 - \frac{k}{\sum_{i=1}^{m}(1+r_i)x_i^2}\right]^{2m-2} - \left[1 - \frac{2k}{\sum_{i=1}^{m}(1+r_i)x_i^2}\right]^{m-1} , \tag{23}$$

$$\sum_{i=1}^{m}(1+r_i)x_i^2 > 2k$$

(iii)Using (19) the UMVUE of $\left(\theta^2\right)^k$ is given by,

$$G_{K,m} = \frac{1}{2^k}\frac{\Gamma m}{\Gamma(m+k)}\left[\sum_{i=1}^{m}(1+r_i)x_i^2\right]^k \tag{24}$$

Special case 3: Substituting $k = \frac{1}{2}$ in (24) we get the UMVUE of θ as,

$$G_{\frac{1}{2},m} = \frac{\Gamma m}{\sqrt{2}\,\Gamma\left(m+\frac{1}{2}\right)}\left[\sum_{i=1}^{m}(1+r_i)x_i^2\right]^{\frac{1}{2}} \tag{25}$$

Since the mode of Rayleigh distribution is θ, the equation given in (25) is UMVUE of mode.

Special case 4: Substituting $k = \frac{-1}{2}$ in (24) we get the UMVUE of $\frac{1}{\theta}$ as,

$$G_{-\frac{1}{2},m} = \frac{\sqrt{2}\,\Gamma m}{\Gamma\left(m-\frac{1}{2}\right)\left[\sum_{i=1}^{m}(1+r_i)x_i^2\right]^{\frac{1}{2}}} \tag{26}$$

Special case 5: The r^th^ moment of the Rayleigh distribution is . $\theta^r\, 2^{\frac{r}{2}}\,\Gamma\left(\frac{r+2}{2}\right)$

Substituting $k=\frac{r}{2}$ in (24) we get UMVUE of r^{th} moment of the Rayleigh distribution as ,

$$\tilde{E}(X^r)=\frac{\Gamma\left(\frac{r+2}{2}\right)\Gamma(m)}{\Gamma\left(m+\frac{r}{2}\right)}\left[\sum_{i=1}^{m}(1+r_i)x_i^2\right]^{\frac{r}{2}} \tag{27}$$

Special case 6: Substituting $r=1$ in (27) we get UMVUE of mean as,

$$\tilde{E}(X)=\frac{\Gamma\left(\frac{3}{2}\right)\Gamma(m)}{\Gamma\left(m+\frac{1}{2}\right)}\left[\sum_{i=1}^{m}(1+r_i)x_i^2\right]^{\frac{1}{2}} \tag{28}$$

Special case 7: The variance of Rayleigh distribution is $\left(2-\frac{\pi}{2}\right)\theta^2$. Its UMVUE is obtained by substituting $k=1$ in (24) as follows,

$$\tilde{Var}[X]=\frac{\left[2-\frac{\pi}{2}\right]\left[\sum_{i=1}^{m}(1+r_i)x_i^2\right]}{2m} \tag{29}$$

Special case 8: Using (24) with $k=-1$ the UMVUE of hazard function $h(x)=\frac{x}{\theta^2}$ can be obtained as ,

$$\tilde{h}(x)=\frac{2x(m-1)}{\left[\sum_{i=1}^{m}(1+r_i)x_i^2\right]} \tag{30}$$

Special case 9: The median of Rayleigh distribution is $\theta(\log 4)^{\frac{1}{2}}$.Using (25) the UMVUE of median is given by ,

$$\tilde{\text{M}}\text{edian}=\frac{\Gamma(m)}{\sqrt{2}\,\Gamma\left(m+\frac{1}{2}\right)}\left[\sum_{i=1}^{m}(1+r_i)x_i^2\right]^{\frac{1}{2}}(\log 4)^{\frac{1}{2}} \tag{31}$$

Special case 10: The p^th^ quantile of Rayleigh distribution is $\xi_p=\theta\sqrt{\log\left(\frac{1}{[1-p]^2}\right)}$.

Using (25) the UMVUE of the p^{th} quantile is given by ,

$$\tilde{\xi}_p = \frac{\Gamma m}{\sqrt{2}\,\Gamma\left(m+\frac{1}{2}\right)}\left[\sum_{i=1}^{m}(1+r_i)x_i^2\right]^{\frac{1}{2}}\sqrt{\log\left(\frac{1}{[1-p]^2}\right)} \tag{32}$$

(iv) Using (24) the UMVUE of the variance of $G_{K,m}$ is given by,

$$\tilde{V}ar[G_{K,m}] = \frac{\left[\sum_{i=1}^{m}(1+r_i)x_i^2\right]^{2k}}{2^{2k}}\left[\left(\frac{\Gamma m}{\Gamma m+k}\right)^2 - \left(\frac{\Gamma m}{\Gamma m+2k}\right)\right] \tag{33}$$

(v) The UMVUE of density $f(x)$ given in (1), for fixed x is given by,

$$\phi_{x,m} = \left(\frac{2x\ (m-1)}{\sum_{i=1}^{m}(1+r_i)x_i^2}\right)\left(1-\frac{x^2}{\sum_{i=1}^{m}(1+r_i)x_i^2}\right)^{m-2}, \tag{34}$$

$$0 < x < \sqrt{\sum_{i=1}^{m}(1+r_i)x_i^2},\ m > 2$$

(vi) The UMVUE of variance of $\phi_{x,m}$, $m > 2$ is given by,

$$\tilde{V}ar[\phi_{x,m}] = \begin{cases} \left(\frac{2x(m-1)}{t}\right)^2\left[1-\frac{x^2}{t}\right]^{2m-4} - \left(\frac{2x(m-1)}{t}\right)\times & t > 2x^2 \\ \left[1-\frac{x^2}{t}\right]^{m-2}\left(\frac{2x(m-2)}{t-x^2}\right)\left[1-\frac{x^2}{t-x^2}\right]^{m-3} & \\ \left(\frac{2x(m-1)}{t}\right)^2\left[1-\frac{x^2}{t}\right]^{2m-4} & x^2 < t \le 2x^2 \\ 0 & \text{otherwise} \end{cases} \tag{35}$$

where $t = \sum_{i=1}^{m}(1+r_i)x_i^2$

.

(vii) Considering x as fixed, the UMVUE of $R\ (x)$ of reliability function, $R(x) = P(X > x)$, $x \ge 0$ is obtained as follows.

Since $R(x) = [h(\theta)]^{x^2}$, where $h(\theta)$ is given in (5) and using (20) with $k = x^2$ the UMVUE $\tilde{R}\ (x)$ of $R\ (x)$ is given by,

$$\tilde{R}(x)=\left[1-\frac{x^2}{\sum_{i=1}^{m}(1+r_i)x_i^2}\right]^{m-1}, \quad 0<x<\sqrt{\sum_{i=1}^{m}(1+r_i)x_i^2} \tag{36}$$

(viii) The UMVUE of the variance of $\tilde{R}(x)$ is given by ,

$$\tilde{V}ar[\tilde{R}(x)]=\begin{cases}\left[1-\frac{x^2}{\sum_{i=1}^{m}(1+r_i)x_i^2}\right]^{2m-2}-\left[1-\frac{2x^2}{\sum_{i=1}^{m}(1+r_i)x_i^2}\right]^{m-1}, & 0<x<\sqrt{\frac{\sum_{i=1}^{m}(1+r_i)x_i^2}{2}}\\ \left[1-\frac{x^2}{\sum_{i=1}^{m}(1+r_i)x_i^2}\right]^{2m-2}, & \sqrt{\frac{\sum_{i=1}^{m}(1+r_i)x_i^2}{2}}<x<\sqrt{\sum_{i=1}^{m}(1+r_i)x_i^2}\\ 0 & \text{otherwise.}\end{cases} \tag{37}$$

(ix) The UMVUE of cumulative distribution function given in (2) is ,

$$\tilde{F}(x)=\begin{cases}0 & , x<0\\ 1-\left(1-\frac{x^2}{\sum_{i=1}^{m}(1+r_i)x_i^2}\right)^{m-1} & , 0<x<\sqrt{\sum_{i=1}^{m}(1+r_i)x_i^2}\\ 1 & \text{, otherwise.}\end{cases} \tag{38}$$

Shanubhouge and Jain [23] have studied the problem of minimum variance unbiased estimation in exponential distribution under progressive type II censored data with binomial removals. They have given the UMVUE for parameter p and various functions of p .Since the joint density $P(\underline{R}=\underline{r})$ given in (9) is independent of θ one gets the same estimators of p and its various functions as given in Shanubhouge and Jain [23].

4 UMVU estimator of $P(X<Y)$

In the following theorem, we derive the UMVUE of $P(X<Y)$. Let m_1 units (out of n_1) on X and m_2 units (out of n_2) on Y are recorded which follow Rayleigh distributions, given in (1) with parameters θ_1 and θ_2 respectively. Let $r_1, r_2, \cdots r_{m_1}$ and $s_1, s_2, \cdots s_{m_2}$ be corresponding removals. We denote,

$$t_1 = \sum_{i=1}^{m_1} (1+r_i) x_i^2 \tag{39}$$

and

$$t_2 = \sum_{i=1}^{m_2} (1+s_i) y_i^2 \ . \tag{40}$$

Theorem 1 : Under progressive Type II censored data the UMVU estimator of $P = P(X<Y)$ for the density given in (1) is given by ,

$$\tilde{P} = \begin{cases} \displaystyle\sum_{j=0}^{m_2-1} (-1)^j \frac{(m_1-1)!(m_2-1)!}{(m_1+j-1)!(m_2-j-1)!} \left(\frac{t_1}{t_2}\right)^j & t_1 < t_2 \\ \displaystyle 1-\sum_{i=0}^{m_1-1} (-1)^i \frac{(m_1-1)!(m_2-1)!}{(m_1-i-1)!(m_2+i-1)!} \left(\frac{t_2}{t_1}\right)^i & t_1 > t_2 \end{cases} \tag{41}$$

where t_1 and t_2 are given by (39) and (40),respectively.

Proof:

We have

$$\tilde{P} = \iint_G \phi_{x,m_1} \phi_{y,m_2} \, dxdy \tag{42}$$

where $G = \left\{ (x,y) : 0 < x < \sqrt{t_1} \ , 0 < y < \sqrt{t_2}, \ x < y \right\}$

Using (34) in (42) and let $t_1 < t_2$, we have,

$$\tilde{P} = \int_0^{\sqrt{t_1}} \int_x^{\sqrt{t_2}} \frac{2x(m_1-1)}{t_1} \frac{2y(m_2-1)}{t_2} \left[1-\frac{x^2}{t_1}\right]^{m_1-2} \left[1-\frac{y^2}{t_2}\right]^{m_2-2} dydx \tag{43}$$

Now

$$\int_x^{\sqrt{t_2}} \frac{2y(m_2-1)}{t_2} \left[1-\frac{y^2}{t_2}\right]^{m_2-2} dy = \left[1-\frac{x^2}{t_2}\right]^{m_2-1} \tag{44}$$

After substituting (44) into (43) we get,

$$= \int_0^{\sqrt{t_1}} \frac{2x(m_1-1)}{t_1}\left[1-\frac{x^2}{t_2}\right]^{m_2-1}\left[1-\frac{x^2}{t_1}\right]^{m_1-2} dx \tag{45}$$

Further simplification of (45) and applying the result $\left(1-w\right)^n = \sum_{j=0}^{n}\binom{n}{j}\left(-1\right)^j w^j$

we get,

$$= (m_1-1)\sum_{j=0}^{m_2-1}\binom{m_2-1}{j}\left(-1\right)^j\left(\frac{t_1}{t_2}\right)^j \int_0^1 w^j(1-w)^{m_1-2}dw \tag{46}$$

Further simplification of (46) gives,

$$\tilde{P} = \sum_{j=0}^{m_2-1}\left(-1\right)^j \frac{\left(m_1-1\right)!\left(m_2-1\right)!}{\left(m_1+j-1\right)!\left(m_2-j-1\right)!}\left(\frac{t_1}{t_2}\right)^j \quad , \quad t_1 < t_2 \tag{47}$$

Similarly we can show that for the case $t_1 > t_2$ the UMVUE of $P(X < Y)$ is,

$$\tilde{P} = 1-\sum_{i=0}^{m_1-1}\left(-1\right)^i \frac{\left(m_1-1\right)!\left(m_2-1\right)!}{\left(m_1-i-1\right)!\left(m_2+i-1\right)!}\left(\frac{t_2}{t_1}\right)^i \qquad t_1 > t_2 \tag{48}$$

5 Illustrative example

In this section we illustrate the use of the estimation methods given in this article.

We consider the data on failure times of 25 ball bearings in endurance test given in Caroni [6]. The data are 17.88, 28.92, 33.00, 41.52, 42.12, 45.60, 48.48, 51.84, 51.96, 54.12, 55.56, 67.80, 67.80, 67.80, 68.64, 68.64, 68.88, 84.12, 93.12, 98.64, 105.12, 105.84, 127.92, 128.04, 173.40.These observations are the number of million revolutions before failure of 25 ball bearings. Raqab and Madi [21], Lee [16], and Lee at el [17], have indicated that a one parameter Rayleigh distribution fits well for these data. We generate a progressive Type II censored data with binomial removals from these data. The progressive censored sample size is $m = 13$.The dropout numbers have been generated using MYSTAT software as follows: r_1 from $B(12, 0.05)$ and $r_i / r_1, r_2, \cdots, r_{i-1}$ have $B(12-\sum_{j=1}^{i-1} r_j, 0.05)$ distribution for $i = 2, 3, \cdots 12$ and set,

$$r_{13} = \begin{cases} 12 - \sum_{j=1}^{12} r_j & \text{if } 12 - \sum_{j=1}^{12} r_j > 0 \\ 0 & \text{o.w.} \end{cases}$$

Table I. Observed times of failure and the dropouts measured in the informative experiment

i	x_i	r_i
1	17.88	1
2	33.0	1
3	42.12	0
4	45.60	1
5	51.84	2
6	55.56	0
7	67.80	0
8	67.80	0
9	67.80	0
10	68.64	0
11	68.64	0
12	68.88	0
13	84.12	7

Using the results given in Section 3, the UMVU estimates of different parametric functions of θ based on data given in Table I are given below.

Table II. The UMVU estimates of different parametric functions of θ based on data given in Table I

Sr. No.	Parametric function	UMVU estimate
1	$\exp\left[\frac{1}{\theta^2}\right]$	$H_{-2,13} = 1.002298$
2	Variance of $H_{-2,13}$	$\tilde{V}ar[H_{-2,13}] = 4.40017E-09$
3	$\exp\left[\frac{-1}{\theta^2}\right]$	$H_{2,13} = 0.99978$
4	Variance of $H_{2,13}$	$\tilde{V}ar[H_{2,13}] = 4.39647E-09$
5	θ	$G_{\frac{1}{2},13} = 63.98665$
6	Variance of $G_{\frac{1}{2},13}$	$\tilde{V}ar[G_{\frac{1}{2},13}] = 76.34856$
7	$\frac{1}{\theta}$	$G_{-\frac{1}{2},13} = 0.015313$
8	Variance of $G_{-1,12}$	$\tilde{V}ar[G_{-\frac{1}{2},13}] = 4.74064E-06$
9	Median	75.3386
10	Third quartile	106.54478
11	Mean	$\tilde{E}(X) = 80.190575$
12	Variance	$\tilde{V}ar[X] = 1721.9759$
13	Hazard function at $x = 80.5$.	$\tilde{h}(x) = 0.018493$
14	Density at $x = 80.5$	$\phi_{80.5,13} = 0.009143$
15	Variance of $\phi_{80.5,13}$	$\tilde{V}ar[\phi_{80.5,13}] = 2.31043E-07$
16	Reliability at $x = 80.5$	$\tilde{R}(x) = 0.4637215$
17	Variance of $\tilde{R}(x)$ at $x = 80.5$	$\tilde{V}ar[\tilde{R}(x)] = 0.011019$
18	c.d.f. at $x = 80.5$	$\tilde{F}(x) = 0.536279$

6 Conclusions:

In this paper we have considered the estimation problem of parameter and its various functions of Rayleigh distribution under the progressive Type II censored data with binomial removals. We have derived an elegant expression for the UMVUE estimators for mode , r^{th} moment , mean , variance , hazard function , median , p^{th} quantile, p.d.f., reliability function and c.d.f. of the Rayleigh distribution. Our method of obtaining these estimators is quite simple than the traditional approach. We have also obtained UMVUE of $P(X < Y)$ by using the UMVUE of p.d.f. These results reduce to Type II censored data (where removals are not allowed) and complete sample case by substituting $r_i = 0$ for $i = 1, 2, \cdots m-1$ and $i = 1, 2, \cdots m.$, respectively.

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