Time Control Charts Using Order Statistics

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Abstract: Control charts are widely used for process monitoring in the manufacturing industry. A recent control scheme based on the cumulative quantity between observations of defects has been proposed which can be easily adopted to monitor the failure process for exponentially distributed inter-failure time. An investigation of its use for reliability monitoring is presented in this paper and the scheme can be easily extended to monitor inter-failure times that follow other distributions. For an effective monitoring of failure process the time between every r^{th} failure (r is a natural number >=2) instead of inter-failure times is considered for developing a variable control chart called Time Control Charts. Use of order statistics is demonstrated for constructing the control limits whereas a similar previous work for exponential model is based on the sampling distribution of cumulative sums of inter-failure times. Our proposed method requires percentiles of the extreme order statistics and relatively simpler than the existing one. Our results are illustrated for some standard life testing models.

1. Introduction

Control charts are based on the principal of monitoring quality variations in the product of a manufacturing process. Though the variations are not completely eliminated, the control chart assures a tolerable zone for the acceptability of the production process with variations. In the classical literature various types of control charts have become popular both in normal and non-normal situations. The concept of applying control chart for monitoring the failure phenomenon is of recent origin. The failure data represented in terms of inter failure times of a product can be used to assess the quality of the product can be used to assess the quality of the product measured in terms of inter failure times. It is but natural to believe that the more the inter failure time, the better the quality of the product. Similarly the less the inter failure time the poorer the quality of the product. However, inter failure time is a positive valued continuous random variable with an inducive probability model for it. Hence, percentiles of the probability model with a specified coverage probability can be explored to design a control chart to monitor the failure mechanism, there by assessing the quality of the product under consideration as tolerable, superior than tolerable and inferior than tolerable. This paper is an attempt to focus this issue using the concept of percentiles of highest order statistics. Similar study is made by Xie et al (2002) with a different approach exemplifying exponential model. Some other research works in this regard are Haworth (1996), Chan et al (2000), Sun et al (2001), Kantam and Ravi Kumar (2010), Kantam and Ravi Kumar (2011), Ramachand Rao et al (2011), Satyaprasad et al (2011a) and Satyaprasad et al (2011b). Our paper takes the same concept and arrives at the control chart through extreme order statistics and works out for any continuous probability model in a simpler and effective way. The rest of the paper is organised as follows: section 2 deals with adoption of highest order statistics for sums of inter failure times, evaluation of the percentiles and the control chart constants. Specific applications in the case of three standard models namely Exponential, Weibull and Rayleigh control charts are also presented.

Section 3 deals with evaluation of the average run length on the basis of Monte Carlo simulation. Comparisons of our limits with those of Xie et al (2002) are also presented. Because of the control chart is basically for inter failure times it is named as time control chart as introduced by Xie et al (2002).

2. Time Control Chart

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n representing n inter failure times of a product governed by the probability model of a continuous random variable X. Let F(x) be the cumulative distribution function of X. These inter failure times can be used for assessing the failure phenomenon with respect to two limits of reference called control limits with a pre specified coverage probability. Following in line with the classical Shewart chart, Xie et al (2002) have taken the coverage probability as 0.9973. The control limits are taken as the equitailed percentiles namely 0.00135 percentile and 0.99865 percentile respectively denoted as T_L, T_U. The graph of (i, X_i) with horizontal parallel lines at T_L, T_U is the time control chart. A point above T_L indicates a larger gap of inter failure times and hence advantage for the production process. A point below T_L is a discouraging phenomenon implying more frequent failures. Points between T_L and T_U, the failure mechanism is tolerable. Thus the time control chart plotted for inter failure times would indicate alarms, advantages and stable failure process. Xie et al (2002) discussed the time control chart for exponential, Weibull distributions.

If r is a natural number (<n), the summations$\sum_{i=1}^{r} X_i$, $\sum_{i=r+1}^{2r} X_i$, $\sum_{i=2r+1}^{3r} X_i$ etc represent the lapse of time consecutively between every r^{th} failure. A control chart for times between every r^{th} failure would through light on the out of control signals than that of inter failure times. Xie et al (2002) named such a control chart as t_r-control chart and developed control limits using the sampling distribution of$\sum_{i=1}^{r} X_i$. They have taken the example of exponential distribution and used the theory that the sum of exponential variates is a gamma variate to get the percentiles of t_r-control chart with the help of cumulative Poisson summations. If the inter failure times are not exponentials, the control limits of t_r-chart of Xie et al (2002) can not be used. Overcoming this drawback we suggest the following alternative approach to get control limits of t_r-chart for any distribution.

If $(X_1, X_2, \ldots, X_r)$; $(X_{r+1}, X_{r+2}, \ldots, X_{2r})$; $(X_{2r+1}, X_{2r+2}, \ldots, X_{3r})$; etc are regarded as independent samples of size r each, i.i.d random variables having F(x) as their common model. $Y_1=X_1$, $Y_2=\sum_{i=1}^{2} X_i$, $Y_3=\sum_{i=1}^{3} X_i$, …, $Y_r=\sum_{i=1}^{r} X_i$ becomes an ordered sample of size r representing the time to first failure, time to second failure,…, time to r^{th} failure respectively. Y_r is the highest order statistics in an ordered sample $Y_1<Y_2<\ldots<Y_r$. Thus, the t_r-chart is the control chart with Y_r as the points on it representing the time to every r^{th} failure. Therefore, when r is fixed the percentiles of highest order statistics in a sample of size r would serve the purpose of control limits for the t_r-chart.

We know that $[F(x)]^r$ is the cumulative distribution function of rth order statistic in a sample of size r for the model F(x). Hence, the percentiles of t_r chart with 0.9973 coverage probability would be the solutions of $[F(x)]^r = 0.99865$ and $[F(x)]^r = 0.00135$. The central line of the tr-chart would be the solution of $[F(x)]^r = 0.5$

In the present paper we have evaluated the control limits of t_r-chart for the standard exponential, half logistic and Rayleigh distributions. For the non standard densities the unknown parameters can be estimated using any classical method of estimation. We have tabulated the control limits for r=2,3,4,5 for the standard models with the respective derivations in each case in the following lines.

Consider the equation $[F(x)]^r = p$. For the substitutions of p = 0.00135, 0.5, 0.99865 in the above equation we will get different values of x named as LCL, CL and UCL respectively. Where F(x) is the cumulative distribution function for exponential, half logistic and Rayleigh distributions. The control chart constants for r = 2,3,4,5 in the models are given in Table 2.1.

Table 2.1 Time Control Chart Constants for Exponential, Half Logistic and Rayleigh distributions using Order Statistics for r=2,3,4,5

r	Exponential			Half-Logistic			Rayleigh		
	LCL	CL	UCL	LCL	CL	UCL	LCL	CL	UCL
2	0.03743	1.22795	7.29341	0.07352	1.76275	7.98656	0.2736	1.56713	3.81927
3	0.11712	1.57843	7.70626	0.22194	2.16271	8.37743	0.48398	1.77675	3.92588
4	0.21279	1.8382	7.98656	0.38817	2.44845	8.67971	0.65238	1.91739	3.99664
5	0.31024	2.04447	8.18072	0.54668	2.67069	8.87387	0.78771	2.02211	4.04493

All these three models are scaled densities involving only an unknown scale parameter say σ for their respective non standard versions. The most popular ML estimate of σ for a given inter failure time data X1, X2, …., Xn for the three models are given by

Exponential: $\hat{\sigma} = \bar{X}$

Half Logistic: $\hat{\sigma} = \bar{X} / \ln(4)$

Rayleigh: $\hat{\sigma} = \sqrt{\frac{1}{2n}\sum_{i=1}^{n} X_i^2}$

These results are illustrated for the example of 60 inter failure times considered by Xie et al (2002). For a ready reference this data is reproduced here.

Failure Number	Time	Failure Number	Time	Failure Number	Time	Failure Number	Time
1	1065.55	16	2932.96	31	35.85	46	239.66
2	535.8	17	987.67	32	362.8	47	93.78
3	540.53	18	1816.18	33	357.85	48	680.45
4	716.2	19	117.21	34	334.48	49	4.83
5	2525.43	20	190.65	35	80.13	50	102.91
6	1264.18	21	943.99	36	1939	51	479.05
7	479.44	22	1084.48	37	77.88	52	156.67
8	1783.22	23	2306.54	38	4.03	53	1286.24
9	473.67	24	6.56	39	98.67	54	443.97
10	2265.42	25	3111.51	40	17.19	55	360.03
11	2191.75	26	283.86	41	289.79	56	414.66

12	1097.26	27	659.39	42	63.99	57	128.9
13	597.59	28	683.48	43	2.46	58	36.1
14	971.16	29	36.14	44	697.68	59	197.31
15	3157.29	30	754.16	45	1167.33	60	418.12

For the sake of comparison with the results of Xie et al (2002) we take r=3 and evaluate the control limits in the above three cases.

The t_3 values are obtained by sum of each non overlapping triple of the above sample in its sequential order. That is, if $X_1, X_2, \ldots, X_{60}$ are the inter failure times, then $Z_1 = \sum_{i=1}^{3} X_i$, $Z_2 = \sum_{i=4}^{6} X_i$, …, $Z_{20} = \sum_{i=58}^{60} X_i$ are the times to 3rd failure of size 20. Control chart for these 20 observations is the t_3-chart with the following control limits.

$$\hat{\sigma} = 768.1847; \text{LCL} = 28.79058; \text{CL} = 1214.104; \text{UCL} = 5609.979$$

Comparison of each Zi is recorded below where A indicates point above UCL, B indicates point below LCL and W indicates point within the control limits.

t_3	Values of t_3	Location w.r.t control limits	t_3	Values of t3	Location w.r.t control limits
Z_1	2141.88	W	Z_{11}	756.5	W
Z_2	4505.81	W	Z_{12}	2353.61	W
Z_3	2736.33	W	Z_{13}	180.58	W
Z_4	5554.43	W	Z_{14}	370.97	W
Z_5	4726.04	W	Z_{15}	1867.47	W
Z_6	5736.81	A	Z_{16}	1013.89	W
Z_7	1251.8	W	Z_{17}	586.79	W
Z_8	3397.58	W	Z_{18}	1886.88	W
Z_9	4054.76	W	Z_{19}	903.59	W
Z_{10}	1473.78	W	Z_{20}	651.53	W

Similar table using the limits of Xie et al (2002) is reproduced below.

t_3	Values of t_3	Location w.r.t control limits	t_3	Values of t3	Location w.r.t control limits
Z_1	2141.88	W	Z_{11}	756.5	W
Z_2	4505.81	W	Z_{12}	2353.61	W
Z_3	2736.33	W	Z_{13}	180.58	A
Z_4	5554.43	W	Z_{14}	370.97	W
Z_5	4726.04	W	Z_{15}	1867.47	W
Z_6	5736.81	W	Z_{16}	1013.89	W
Z_7	1251.8	W	Z_{17}	586.79	W
Z_8	3397.58	W	Z_{18}	1886.88	W

Z_9	4054.76	W	Z_{19}	903.59	W
Z_{10}	1473.78	W	Z_{20}	651.53	W

The first out of control signal with our control limits is observed at the 6^{th} time to third failure. Whereas, the control limits of Xie et al (2002) it is observed at the 13^{th} time to third failure. In other words, the run length required to detect an out of control signal with our limits is less than that of the control limits of Xie et al (2002). This is an indication that our control limits may have a preference w.r.t the ARL concept. This investigation is in progress by the authors. However, as a preliminary study a Monte Carlo simulation is made on the basis of 1000 runs of t_3 control charts with each chart for 20 points on the plot. The methodology and findings are presented in section 3.

3. Preliminary Simulation Study

10,000 random observations are generated from exponential distribution. First they are grouped into 60 observations are used to get 20 points on a t_3 chart. In all we get 1600 such t_3 charts. The respective control limits are obtained as narrated in section 2. The number of t_3 points preceding the first out of control fall out for each of the 1600 charts is counted. In this study only 40 charts have an out of control signal. For the remaining charts all the points are within the control limits. The average of the counts of points preceding the first out of control fall out over the 40 charts is considered as an indicator of average run length (ARL), which in our case is worked out to be 8. This is less than 12, the corresponding number of Xie et al (2002). Thus, our approach of control limits for a t_r-chart may have less ARL than in the approach of Xie et al (2002). Further evaluation of control limits in our approach is simpler involving inversion of r^{th} power of the distribution function of the parent population or statistical model under consideration.

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