An Iterative Algorithm Using the Statistical Perspective of Bias for Efficient Polynomial Approximation by Modified MKZ Operator.

Robin Antoine, *Ashok Sahai

Department of Mathematics & Computer Science,

The University of West Indies; St. Augustine Campus @ TRINIDAD.

Abstract

This paper aims at constructing an iterative computerizable numerical algorithm for an improved polynomial approximation by a modified version of 'MKZ' operator. The algorithm uses the 'Statistical Perspective of Bias' for exploiting the information about the unknown function 'f' available in terms of its known values at the 'pre-chosen-knots' in C [0, 1/2] more fully with the proposed modified operator. The improvement, achieved by an a-posteriori use of this information, happens iteratively. Any typical iteration uses the typical concepts of 'Bias'. The potential of the achievable efficiency through the proposed 'computerizable numerical iterative algorithm' is illustrated per an 'empirical study' for which the function 'f' is assumed to be known in the sense of simulation. The illustration has been confined to "Three Iterations" only, for the sake of simplicity of illustration.

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***Corresponding Author's Email: sahai.ashok@gmail.com**

1. Introduction.

Polynomial approximation, Real analysis, and numerical analysis & computing have been studied in the various interestingly mutually gainful contexts. To get the significance of various polynomial approximation operators in these contexts one does well to browse-n-study the useful contents in the references [1] to [5], [7],[8], [10] and [11].

W. Meyer- Konig and K. Zeller (1960) proposed the generalization of the well-known Bernstein's polynomial approximation operator. The **Meyer-Konig and Zeller** operators **MKZn (f; x) [12]**, are defined as:

$$\mathbf{MKZ_n\ (f;\ x)} = \sum_{k=0}^{k=\infty} \mathbf{m_{n,k}\,(x)}\,.\,f\left(\frac{k}{n+k}\right);\ \mathbf{f\ \varepsilon\ C\ [0,\ 1)\ \&\ x\ \varepsilon\ C\ [0,\ 1).} \qquad \mathbf{(1.1)}$$

Wherein; $\mathbf{m_{n,\,k}(x)} = \binom{n+k-1}{k} x^k (1-x)^{n-1}$.

Motivated by that modification by **Heinz-Gerd Lehnhoff** (1984) **[6]**, but slightly different, we propose a more appropriate **(Finite-Terms)** modification of the **Meyer-Konig and Zeller** polynomial approximation operator. **We define our "Modified Meyer-Konig and Zeller"** operators **MMKZn (f; x) as follows: →**

$$\mathbf{MMKZ_n\ (f;\ x)} = \frac{\sum_{k=0}^{k=n} \mathbf{m_{n.k}(x)}\,.\,f\left(\frac{k}{n+k}\right)}{\mathbf{T_n(x)}};\ \mathbf{f\ \varepsilon\ C\ [0,\ ½]\ \&\ x\ \varepsilon\ C\ [0,\ ½].} \qquad \mathbf{(1.2)}$$

Wherein, $\mathbf{T_n\,(x)} = \sum_{k=0}^{k=n} \mathbf{m_{n,\,k}(x)}$.

This modification is apparently a more-appropriate one inasmuch as **"(MMKZ$_n$ (f))(x)"** could well be interpreted as **"Weighted-Average"** of the used **(n +1)** known values of the unknown function **"f(x)"** , namely **'f(k/n)' ; k = 0 (1) n**. The **"Weights"** being "**m$_{n,\,k}$(x)**"; "**m$_{n,\,k}$(x)'s"** could be interpreted as **"probabilities"** ["**m$_{n,\,k}$(x)>0"]**. As such, therefore,

$$\mathbf{(MMKZ_n\ (f))(x) = E\ (f\ (x))} \qquad \mathbf{(1.3)}$$

Incidentally, **as we could use a suitable transformation (translation-n-change-of-scale) of the variable 'x'**, we could assume, without loss of generality, that we are interested in the approximation of a bounded function **f ε C[0, 1/2], even if the impugned function could, originally, be rather a bounded one f**$\in C[a, b]$.

2. **Reduced-Bias Iterative Improvement Algorithm for Modified Meyer-Konig and Zeller operators.**

In this section we propose the **"Reduced-Bias Iterative Improvement Algorithm for Modified Meyer-Konig and Zeller operators"** using the **Statistical Perspectives of 'Bias'**. This is analogous to that in the paper of **Sahai** (2004) **[10] for the "Bernstein's Polynomial Approximation operators".** In the statistical sense **'($MMKZ_n$ (f)) (x)'** is an estimate of the unknown function **'f(x); say ~ μ.**

Now, we use our **estimator/**modified **Meyer-Konig and Zeller** Operators **'($MMKZ_n$ (f)) (x)' to estimate the values of the unknown function 'f(x)' at the knots '(k/(k+n))', say Et f(k/(k+n)), k = 0 (1) n, using known values of the unknown function "f(x)", namely "f(k/(k+n))'s"; k = 0 (1) n**. Hence the **"Knot-Wise Error", say Er f (k/(k+n)) ≡ Et f(k/(k+n)) – f(k/(k+n)); k = 0 (1) n could be generated to lead to the calibration of the "Bias Polynomial Error Function",**

$$\text{Say Er } MMKZ_n\,(f;\,x) = \frac{\sum_{k=0}^{k=n} m_{n,k}(x).\text{Er } f\left(\frac{k}{k+n}\right)}{\sum_{k=0}^{k=n} m_{n,k}(x)} \tag{2.1}$$

On the other hand the **"Modified Meyer-Konig and Zeller" approximation/estimator of the unknown function 'f(x)' is "($MMKZ_n$ (f)) (x)", as per the equation (1.3) in the preceding section. This enables us to achieve "Iteration # 1" of the iterative algorithm, the "Reduced-Bias Polynomial" approximation/ estimator of the unknown function "f(x)" just by subtracting the "Estimated Bias Error Polynomial" per (2.1) above to get:**

$$\text{Say I [\#1] } MMKZ_n\,(f;\,x) = (MMKZ_n)\,(x) - \text{Er } MMKZ_n\,(f;\,x) \tag{2.2}$$

This completes the **"First Iteration"**!

We again apply the details of the **"Iteration # 1"** on **I [#1] $MMKZ_n$ (f; x), exactly similarly to the aforesaid on "$MMKZ_n$ (f; x)" to obtain our "Reduced-Bias Modified Meyer-Konig and Zeller Approximation/ Estimator Polynomial, Say:**

I [#2] ($MMKZ_n$ (f)) (x) ≡ [Reduced- Bias Version Using 'Bias-Reduction' on Reduced-Bias Modified Meyer-Konig and Zeller (Iteration #1) Polynomial, namely I [#1] $MMKZ_n$ (f; x)] (2.3)

As such, we could continue doing so for any number of iterations 'J', using our 'Reduced-Bias Iterative Algorithm on the preceding Reduced-Biased I [# J-1] $MMKZ_n$ (f; x), as long as we please!

3. The empirical simulation study.

To illustrate the **gain in the efficiency** in our proposed **"Modified MKZ Operators"** by using our proposed **"Iterative Bias-Reduction Algorithm of Improvement of Polynomial Approximation Operator"**, we have carried out an empirical study. We have taken the **example-cases of n = 2, 3, and 4 (i.e. n + 1 = 3, 4 and 5 knots)** in the empirical study to numerically illustrate the **relative gain in efficiency in using the Algorithm Vis-à-Vis the Original Modified MKZ Polynomial Operator in each example-case of the n-values**. Essentially, the empirical study is a simulation one wherein we would have to assume that the function, being tried to be approximated, namely **f(x)** being known to us. Once again we have confined to the illustrations of the relative gain in efficiency by the Iterative Improvement for the following four illustrative functions:
f (x) = exp(x), ln (2 + x), sin (2 + x), and 2^x.

To illustrate the **POTENTIAL** of the improvement with our proposed **Reduced-Bias-Iterative Algorithm**, we have considered **THREE Iterations**, and the numerical values of **seven – quantities; with the three Percentage Relative Errors (PREs)** corresponding to our **Improvement Iteration (# = 1, 2, or 3) (PRE_I (#) $MMKZ_n$ (f ;x)) [n]), Original Modified MKZ Polynomial Operator (PRE_$MMKZ_n$ (f; x) [n]), and the three corresponding Percentage Relative Gains (PRGs)** in using our Iterative Algorithmic **Modified MKZ** Polynomial Operators in place of the Original Modified **MKZ** Polynomial Operators **$MMKZ_n$ (f ; x) [n], namely (PRG_I (#)$MMKZ_n$ (f ; x) [n]; # = 1 (1) 3).** These quantities are defined as follows. The **PRE using (original) Modified MKZ (Polynomial) using n intervals in [0, 1/2], i.e. [(k- 1)/n, k/n]; k = 1 (1) n:**

$$\textbf{PRE\_ MMKZ}_n\textbf{ (f; x) [n]} = \frac{\left|\int_0^{1/2} f(x)dx - \int_0^{1/2} \mathbf{MMKZ}_n(f;x)dx\right|}{\int_0^{1/2} f(x)dx} \times 100.$$

The PRE using Improvement Iteration (I #1, or 2, or 3) on Meyer-Konig and Zeller (Polynomial) Operator using n intervals in [0, 1], i.e. [(k-1)/n, k/n]; k = 1(1) n:

$$\textbf{PRE\_ I (\#) MMKZn (f; x) [n]} = \frac{\left|\int_0^{1/2} f(x)dx - \int_0^{1/2} \mathbf{I(\#)\,MMKZ}_n(f;x)dx\right|}{\int_0^{1/2} f(x)dx} \times 100;\ \# = 1 \text{ or } 2 \text{ or} 3.$$

The **PREs** respective to the **Original Modified MKZ Polynomial Operator and respective to the First, Second and Third Algorithmic Improvement Iteration Polynomials, respectively, for each of the example number of approximation Knots/ Intervals and the Percentage Relative Gains (PRGs), defined exactly analogously to PREs, by using the proposed Reduced-Bias Iterative Algorithmic Improvement Iteration: I(#) (e.g.; 1, or 2, or 3) Polynomials with the n intervals in [0, 1/2] over using the (Original) Modified MKZ Polynomial Operator for the approximation of the (Targeted) function, f(x), are tabulated in the following four tables ~ Tables 1 to Table 4 in the "APPENDIX".**

4. Conclusion.

The tabulated values of PRGs in the "APPENDIX" amply illustrate the 'Relative Gains' by using the proposed "Iterative Algorithm using the Statistical Perspective of BIAS for efficient polynomial approximation by the Modified MKZ Operator". Even for 5 knots (n = 4), the PRGs are above 98% for all example-functions, after only THREE iterations! For f(x) = exp(x) & 2^x it is above 99%, after the third iteration!!

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APPENDIX.

Table 1 :(Iterative) Algorithmic (In %) Relative (Absolute) Efficiency/ Gain for f (x) = exp(x).

Items ↓	n→ 2	3	4
PRE_MMKZ$_n$ (f; x) [n]	9.12837310	6.93757277	5.61323912
PRE_I (1) MMKZ$_n$ (f; x) [n]	3.03976880	1.78943420	1.22035583
PRE_I (2) MMKZ$_n$ (f; x) [n]	0.68259688	0.20929390	0.16491064
PRE_I (3) MMKZ$_n$ (f; x) [n]	0.40586338	0.20794679	0.04428672
PRG_I (1) MMKZ$_n$ (f; x) [n]	66.69977479	74.20662443	78.25932923
PRG_I (2) MMKZ$_n$ (f; x) [n]	92.52225036	96.98318252	97.06211269
PRG_I (3) MMKZ$_n$ (f; x) [n]	95.55382565	97.00260015	99.21103087

Table 2: (Iterative) Algorithmic (In %) Relative (Absolute) Efficiency/Gain for f (x) = ln (2+x).

Items ↓	n→ 2	3	4
PRE_MMKZ$_n$ (f; x) [n]	6.11501420	4.56705957	3.63763478
PRE_I (1) MMKZ$_n$ (f; x) [n]	2.46560627	1.38417537	0.89150476
PRE_I (2) MMKZ$_n$ (f; x) [n]	0.85359707	0.36627566	0.20192908
PRE_I (3) MMKZ$_n$ (f; x) [n]	0.15478757	0.05995129	0.04147832
PRG_I (1) MMKZ$_n$ (f; x) [n]	59.67946767	69.69219804	75.49218617
PRG_I (2) MMKZ$_n$ (f; x) [n]	86.04096336	91.98005506	94.44889050
PRG_I (3) MMKZ$_n$ (f; x) [n]	97.46872913	98.68731086	98.85974474

Table 3: (Iterative) Algorithmic (In %) Relative (Absolute) Efficiency/Gain for f (x) = sin (2+x).

Items ↓	n→ 2	3	4
PRE_MMKZ$_n$ (f; x) [n]	7.06507830	5.39037208	4.38029781
PRE_I (1) MMKZ$_n$ (f; x) [n]	2.20579731	1.31316470	0.91349603
PRE_I (2) MMKZ$_n$ (f; x) [n]	0.74952621	0.24428022	0.09027958
PRE_I (3) MMKZ$_n$ (f; x) [n]	0.11015666	0.07289120	0.06485843
PRG_I (1) MMKZ$_n$ (f; x) [n]	68.77886958	75.63870025	79.14534421
PRG_I (2) MMKZ$_n$ (f; x) [n]	89.39111236	95.46821217	97.93896238
PRG_I (3) MMKZ$_n$ (f; x) [n]	98.44082886	98.64775185	98.51931440

Table 4: (Iterative) Algorithmic (In %) Relative (Absolute) Efficiency/Gain for f (x) = 2^x.

Items ↓	n→ 2	3	4
PRE_MMKZ$_n$ (f; x) [n]	6.60776779	5.00068830	4.03151819
PRE_I (1) MMKZ$_n$ (f; x) [n]	2.30795557	1.34039319	0.90094768
PRE_I (2) MMKZ$_n$ (f; x) [n]	0.43874828	0.20619847	0.14124729
PRE_I (3) MMKZ$_n$ (f; x) [n]	0.34265060	0.10310347	0.01530829
PRG_I (1) MMKZ$_n$ (f; x) [n]	65.07208420	73.19582595	77.65239679
PRG_I (2) MMKZ$_n$ (f; x) [n]	93.36011345	95.87659809	96.49642432
PRG_I (3) MMKZ$_n$ (f; x) [n]	94.81442722	97.93821439	99.62028461