On the Double Prior Selection for the Parameter of Poisson Distribution

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Abstract

This paper provides a comparison of double informative priors which are assumed for an unknown parameter of Poisson distribution. The idea is that some time for a single true unknown parameter, different prior information are available; usually we use one informative prior to incorporate that prior knowledge and ignoring the other information. So to include two different kind of information in the analysis, two different priors have been selected for a single unknown parameter of Poisson distribution. We have assumed three double priors: Gamma-Chi-square distribution, Gamma-Exponential distribution, Chi-square-Exponential distribution and one as usual prior: Gamma distribution for the unknown parameter of the Poisson model. The comparison is based upon the posterior variances, posterior predictive variances and the posterior predictive probabilities. An application to real life data is also introduced to illustrate the method. Finally a suitable double prior for the unknown parameter of a Poisson distribution is suggested.

Keywords: Density kernel, Informative prior, Posterior distribution, Posterior predictive distribution, Predictive intervals, Predictive probabilities, Prior distribution.

1. Introduction

The Poisson distribution provides a realistic model for many random phenomena. Since the values of a Poisson random variable are the nonnegative integers, any random phenomenon for which a count of some sort is of interest, is a candidate for modeling by assuming a Poisson distribution, it is an important family of discrete probability distributions.

The handbook by Haight (1967) on Poisson distribution gives many references and its applications in variety of fields. Howlader and Balasooriya (2003) derived Bayes estimators for distribution function of Poisson distribution based upon complete and truncated data by using Gamma prior. Under complete data both maximum likelihood and Bayes estimators are compared through Monte Carlo study. Bolstad (2007, p. 183) discussed the use of uniform prior, Jeffreys prior and Gamma prior for unknown parameter of Poisson distribution. Aslam and Haq (2009) made comparison of informative priors for unknown variance of Normal distribution, by using posterior variances, posterior skewness and posterior predictive variances for different sets of hyperparameters. Saleem and Aslam (2008) selected a suitable prior for the mixture of Rayleigh distribution using the predictive intervals. The prior is selected on the basis of shortest Bayesian predictive interval, further these intervals are evaluated for different choices of hyperparameters. Sinha (1998) has obtained 95% predictive intervals for various sets of values of hyperparameters using the sample size n=100. Tahir and Hussain (2008) made comparison of two non-informative priors using Bayes point estimates, posterior variances and posterior coefficient of skewness for Poisson distribution. For further details see (Congdon (2006, p. 279), Irony (1992) and Sahai and Khurshid (1993)).

In this paper, the posterior distributions for the unknown parameter of Poisson distribution are derived using the following four priors:

(i). Gamma-Chi-square distribution.

(ii). Gamma-Exponential distribution.

(iii). Chi-square-Exponential distribution.

(iv). Gamma distribution.

The posterior predictive distributions under these informative priors have also been developed. For the selection of the suitable prior, we have compared the informative priors on the basis of their posterior variances, posterior predictive variances and predictive intervals computed from the posterior predictive distribution.

The organization of paper is as follows.

Sections 2 and 4 include the posterior and posterior predictive distributions which are derived under these four informative priors. Section 3 provides the simulation study in which the assumed informative priors are compared. In Section 5, a real life example is discussed in detail. Section 6 finally provides the conclusion.

2. The Posterior distributions of the unknown parameter of Poisson distribution

The posterior distributions using three double informative priors for the unknown parameter of Poisson distribution are derived in the following sections:

Let $X_1,...,X_n$ be a random sample drawn from the Poisson distribution having unknown parameter λ. The likelihood function of the sample observations: $\mathbf{x}: x_1, x_2, ..., x_n$ is:

$$L(\mathbf{x};\lambda) = \prod_{i=1}^{n} f(x_i;\lambda) = \frac{e^{-n\lambda}\lambda^{\sum_{i=1}^{n} x_i}}{\prod_{i=1}^{n}(x_i)!}, \qquad x_i = 0,1,2,\ldots,\infty, \qquad (2.1)$$

where $\lambda>0$ is unknown parameter.

2.1. Gamma-Chi-squared distribution as a double prior

It is assumed that the prior distribution of λ is Gamma distribution with hyperparameters 'a_1' and 'b_1' which is given below;

$$f_{11}(\lambda) = \frac{b_1^{a_1}}{\Gamma(a_1)} \lambda^{a_1-1} e^{-\lambda b_1}, \qquad \lambda>0,\ a_1>0,\ b_1>0. \qquad (2.1.1)$$

Similarly, the second prior distribution is assumed to be the Chi-squared distribution with hyperparameter 'c_1'. The p.d.f of this prior is;

$$f_{12}(\lambda)=\frac{1}{\Gamma\left(\frac{c_1}{2}\right)2^{\frac{c_1}{2}}}\ \lambda^{\frac{c_1}{2}-1}\ e^{-\frac{\lambda}{2}}, \qquad \lambda>0,\ c_1>0. \qquad (2.1.2)$$

Now we define the double prior for λ by combining these two priors which is as follows.

$$f_1(\lambda)\ \propto\ f_{11}(\lambda)\ f_{12}(\lambda), \qquad (2.1.3)$$

$$f_1(\lambda)=k\left(\lambda^{a_1-1}\ e^{-\lambda b_1}\ \lambda^{\frac{c_1}{2}-1}\ e^{-\frac{\lambda}{2}}\right), \qquad (2.1.4)$$

where $k=\frac{\left(b_1+0.5\right)^{a_1+0.5c_1}}{\Gamma\left(a_1+0.5c_1\right)}$.

Now the posterior distribution of λ for the given data $'\mathbf{x}'$ is:

$$p_1\left(\lambda\,|\,\mathbf{x}\right)\ \propto\ L\left(\mathbf{x};\lambda\right)\ f_1\left(\lambda\right), \qquad (2.1.5)$$

$$p_1\left(\lambda\,|\,\mathbf{x}\right)\ \propto\ e^{-n\lambda}\lambda^{\sum_{i=1}^{n}x_i}\ \lambda^{a_1+\frac{c_1}{2}-1-1}\ e^{-\lambda(b_1+\frac{1}{2})}, \qquad (2.1.6)$$

$$p_1\left(\lambda\,|\,\mathbf{x}\right)\ \propto\ \lambda^{\left(a_1+\frac{c_1}{2}+\sum_{i=1}^{n}x_i-1\right)-1}\ e^{-\lambda\left(n+b_1+\frac{1}{2}\right)}, \qquad \lambda>0, \qquad (2.1.7)$$

which is the density kernel of the Gamma distribution with parameters: $\alpha_1=a_1+\frac{c_1}{2}+\sum_{i=1}^{n}x_i-1$ and $\beta_1=n+b_1+\frac{1}{2}$. Hence the posterior distribution of λ for given data is Gamma distribution having parameters α_1 and β_1.

2.2. Gamma-Exponential distribution as a double prior

It is assumed that the double prior distribution of λ is Gamma distribution with hyperparameters 'a_2' and 'b_2' and the Exponential distribution with hyperparameter 'c_2', which is given below;

$$f_2(\lambda)\ \propto\ \lambda^{a_2-1}\ e^{-\lambda(b_2+c_2)}, \qquad \lambda>0. \qquad (2.2.1)$$

Now the posterior distribution of λ for the given data $'\mathbf{x}'$ is:

$$p_2\left(\lambda\,|\,\mathbf{x}\right) \;\propto\; \lambda^{\left(a_2+\sum_{i=1}^{n}x_i\right)-1}\, e^{-\lambda(b_2+c_2+n)}, \qquad \lambda>0, \qquad (2.2.2)$$

which is the density kernel of the Gamma distribution with parameters: $\alpha_2 = a_2 + \sum_{i=1}^{n} x_i$ and $\beta_2 = b_2 + c_2 + n$. Thus the posterior distribution of λ for given data is a Gamma distribution with parameters α_2 and β_2.

2.3. Chi-square-Exponential distribution as a double prior

Now it is assumed that the double prior distribution of λ is Chi-square distribution with hyperparameter 'a_3' and the Exponential distribution with hyperparameter 'c_3', which is given below;

$$f_3(\lambda) \propto \lambda^{\frac{a_3}{2}-1}\, e^{-\lambda(c_3+\frac{1}{2})}, \qquad \lambda>0. \qquad (2.3.1)$$

Now the posterior distribution of λ for the given data $'\mathbf{x}'$ is:

$$p_3\left(\lambda\,|\,\mathbf{x}\right) \;\propto\; \lambda^{\left(\frac{a_3}{2}+\sum_{i=1}^{n}x_i\right)-1}\, e^{-\lambda\left(c_3+\frac{1}{2}+n\right)}, \qquad \lambda>0, \qquad (2.3.2)$$

which is the density kernel of the Gamma distribution with parameters: $\alpha_3 = \frac{a_3}{2} + \sum_{i=1}^{n} x_i$ and $\beta_3 = c_3 + \frac{1}{2} + n$. So the posterior distribution of λ for given data is Gamma distribution having parameters α_3 and β_3.

2.4. Gamma distribution as prior

It is usually assumed that the single prior distribution of λ is Gamma distribution with hyperparameters 'a_4' and 'b_4' which is given below;

$$f_4(\lambda)=\frac{b_4^{\,a_4}}{\Gamma\left(a_4\right)}\;\lambda^{a_4-1}\;e^{-\lambda b_4}, \qquad \lambda>0,\; a_4>0,\; b_4>0. \qquad (2.4.1)$$

Now the posterior distribution of λ for the given data $\mathbf{'x'}$ is:

$$p_4\left(\lambda\,|\,\mathbf{x}\right)\;\propto\;\lambda^{\left(a_4+\sum_{i=1}^{n}x_i\right)-1}\;e^{-\lambda(n+b_4)}, \qquad \lambda>0, \qquad (2.4.2)$$

which is the density kernel of the Gamma distribution with parameters: $\alpha_4=a_4+\sum_{i=1}^{n}x_i$ and $\beta_4=n+b_4$. So the posterior distribution of λ for given data is Gamma distribution having parameters α_4 and β_4. The Gamma distribution is a natural conjugate prior for λ of Poisson distribution (see Gelman et. al. (1995) and Bernardo and Smith (1994)).

3. Simulation study

In this section, simulation study is conducted to investigate the performance of the assumed informative priors. The random samples of sizes n=50, 80 and 100 are generated from Poisson distribution with parameter λ values 2, 5, 8 and 10. Based upon 10,000 simulations the results of posterior and posterior predictive variances have been made. R and Mathematica have been used to carry out the results.

3.1. Comparison of priors with respect to posterior variances

The variances of the posterior distributions under all of assumed informative priors are calculated by assuming different sets of values of hyperparameters, which are given in Tables 3.1.1, 3.1.2 and 3.1.3 (see appendix).

The posterior variances under the double prior Gamma-Exponential distribution are less as compare to other informative priors, which show that this prior is efficient as compare to other priors and this less variation in posterior distribution helps in making more precise Bayesian estimation of the true unknown parameter λ of Poisson distribution.

4. The Posterior predictive distributions

As we have observed that the there is only one type of posterior distribution derived under all the priors i.e. Gamma distribution. We now derive posterior predictive distribution under this posterior distribution.

4.1. The Posterior predictive distribution under Gamma-Chi-square prior

The posterior predictive distribution for $Y = X_{n+1}$ given $\mathbf{x}: x_1, x_2, ..., x_n$ under Gamma-Chi-square distribution is:

$$p_1\left(y|\mathbf{x}\right)=\int_0^\infty p\left(y|\lambda\right)p_1\left(\lambda|\mathbf{x}\right)\,d\lambda\,, \qquad (4.1.1)$$

$$p_1\left(y|\mathbf{x}\right)=\int_0^\infty \frac{e^{-\lambda}\lambda^y}{y!}\frac{\beta_1^{\alpha_1}}{\Gamma\left(\alpha_1\right)}\,\lambda^{\alpha_1-1}e^{-\beta_1\lambda}\,d\lambda\,, \qquad (4.1.2)$$

$$p_1\left(y\,|\mathbf{x}\right)=\frac{\beta_1^{\alpha_1}\Gamma\left(\alpha_1+y\right)}{\Gamma\left(\alpha_1\right)\,\mathrm{y}!\,\left(\beta_1+1\right)^{\alpha_1+y}}\,, \qquad y=0,1,2,\ldots. \qquad (4.1.3)$$

Which is the probability mass function of Poisson-Gamma-distribution i.e.

$$Y|\mathbf{x}\sim Pg\left(\alpha_1,\beta_1,1\right), \qquad \alpha_1>0,\beta_1>0,y=0,1,2,\ldots. \qquad (4.1.4)$$

Where α_1 and β_1 are given in (2.1.7). The Poisson-Gamma-distribution is defined in appendix I.

4.2. The Posterior predictive distribution under Gamma-Exponential prior

The posterior predictive distribution for $Y = X_{n+1}$ given $\mathbf{x}: x_1, x_2, ..., x_n$ under Gamma-Exponential prior:

$$p_2\left(y|\mathbf{x}\right)=\int_0^\infty \frac{e^{-\lambda}\lambda^y}{y!}\frac{\beta_2^{\alpha_2}}{\Gamma\left(\alpha_2\right)}\,\lambda^{\alpha_2-1}e^{-\beta_2\lambda}\,d\lambda\,, \qquad (4.2.1)$$

$$p_2\left(y\,|\mathbf{x}\right)=\frac{\beta_2^{\alpha_2}\Gamma\left(\alpha_2+y\right)}{\Gamma\left(\alpha_2\right)\,\mathrm{y}!\,\left(\beta_2+1\right)^{\alpha_2+y}}\,, \qquad y=0,1,2,\ldots. \qquad (4.2.2)$$

Which is the probability mass function of Poisson-Gamma-distribution i.e.

$$Y \mid \mathbf{x} \sim Pg(\alpha_2, \beta_2, 1), \qquad \alpha_2 > 0, \beta_2 > 0, y = 0,1,2,\dots. \qquad (4.2.3)$$

Where α_2 and β_2 are given in (2.2.2).

4.3. The Posterior predictive distribution under Chi-square-Exponential prior

The posterior predictive distribution for $Y = X_{n+1}$ given $\mathbf{x}: x_1, x_2, \dots, x_n$ under Chi-square-Exponential prior:

$$p_3(y \mid \mathbf{x}) = \int_0^\infty \frac{e^{-\lambda}\lambda^y}{y!} \frac{\beta_3^{\alpha_3}}{\Gamma(\alpha_3)} \lambda^{\alpha_3 - 1} e^{-\beta_3 \lambda}\, d\lambda, \qquad (4.3.1)$$

$$p_3(y \mid \mathbf{x}) = \frac{\beta_3^{\alpha_3}\Gamma(\alpha_3 + y)}{\Gamma(\alpha_3)\ y!\ (\beta_3 + 1)^{\alpha_3 + y}}, \qquad y = 0,1,2,\dots. \qquad (4.3.2)$$

Which is also the probability density function of Poisson-Gamma-distribution i.e.

$$Y \mid \mathbf{x} \sim Pg(\alpha_3, \beta_3, 1), \qquad \alpha_3 > 0, \beta_3 > 0, y = 0,1,2,\dots, \qquad (4.3.3)$$

Where α_3 and β_3 are given in (2.3.2).

4.4. The Posterior predictive distribution under Gamma prior

The posterior predictive distribution for $Y = X_{n+1}$ given $\mathbf{x}: x_1, x_2, \dots, x_n$ under Gamma prior is:

$$p_4(y \mid \mathbf{x}) = \int_0^\infty \frac{e^{-\lambda}\lambda^y}{y!} \frac{\beta_4^{\alpha_4}}{\Gamma(\alpha_4)} \lambda^{\alpha_4 - 1} e^{-\beta_4 \lambda}\, d\lambda, \qquad (4.4.1)$$

$$p_4(y \mid \mathbf{x}) = \frac{\beta_4^{\alpha_4}\Gamma(\alpha_4 + y)}{\Gamma(\alpha_4)\ y!\ (\beta_4 + 1)^{\alpha_4 + y}}, \qquad y = 0,1,2,\dots. \qquad (4.4.2)$$

Which is the probability mass function of Poisson-Gamma-distribution i.e.

$$Y \mid \mathbf{x} \sim Pg(\alpha_4, \beta_4, 1), \qquad \alpha_4 > 0, \beta_4 > 0, y = 0,1,2,\dots. \qquad (4.4.3)$$

Where α_4 and β_4 are given in (2.4.2).

4.5. Comparison of priors using the posterior predictive variances

The posterior predictive variances using different prior distributions are given in the Tables 4.5.1, 4.5.2 and 4.5.3 (see appendix).

1. For the posterior predictive (Poisson-Gamma) distribution, we have

$$\text{Var (y|}\mathbf{x}\text{)} = \frac{\alpha_i}{\beta_i}\left(1+\frac{1}{\beta_i}\right) \text{ for } i\text{=1,2,3,4.} \tag{4.5.1}$$

In the Tables 4.5.1, 4.5.2 and 4.5.3, it is observed that the values of the posterior predictive variances computed under the double prior Gamma-Exponential distribution using different values of hyperparameters are less as compare to other priors, which means we can prefer the prior Gamma-Exponential distribution as a suitable double prior for the unknown parameter λ of Poisson distribution. Further this less variation in the posterior predictive distribution will help us in closely estimating the true probabilities of the future observations.

4.6. Construction of the predictive probabilities using priors

As the predictive distribution here is a discrete probability (Poisson-Gamma) distribution due to which it is very much difficult to compute the exact $(1-\alpha)100\%$ i.e. 95% or 99% predictive intervals, so to over this problem, a fixed predictive interval is chosen for the discrete random variable '*Y*' which is [L,U]=[0,5], where 'L' is the lower limit and the 'U' is the upper limit. Then, the predictive probabilities are computed under this fixed interval from the predictive distributions, derived under the assumed priors. As $(1-\alpha)100\%$ Bayesian predictive interval [L,U] is given by

$$\sum_{y=L}^{y=U} p(y\,|\,\mathbf{x}) = (1-\alpha)100\%, \qquad \text{for} \quad y=0,1,2,\ldots. \tag{4.6.1}$$

Also $p(y\,|\,\mathbf{x})$ is given above for each prior.

4.7. Comparison of the predictive probabilities for the fixed predictive interval

The values of the predictive probabilities are computed for different values of the hyperparameters. In order to compute posterior predictive probabilities for the fixed Bayesian predictive intervals i.e. [0,5] and [0,10], we have taken $\sum_{i=1}^{100} x_i$ =200.0884 and $\sum_{i=1}^{100} x_i$ =1000.298 as fixed quantities and are calculated after 10,000 simulations from Poisson distribution with parameters λ=2 and λ=10 respectively.

On observing the Tables 4.7.1 and 4.7.2 (see appendix), in the 4th column the predictive probabilities computed from the posterior predictive distribution under the second double prior (Gamma-Exponential distribution) and third double prior (Chi-square-Exponential distribution) are greater as compared to the other two priors, which shows that by using these two priors especially Gamma-Exponential distribution as a combine prior for the unknown parameter λ of Poisson distribution, we have the predictive intervals [0,5] and [0,10] with the maximum probability under the assumed values of the hyperparameters (subject to the availability of the prior information). Further, this can be interpreted as for the prediction of the future observation '*Y*' with predicted probability of 99.938%, the Bayesian predictive interval i.e. [0,5] is the shortest possible under the double prior Gamma-Exponential distribution as compared to all other priors. Similarly one can see in Table 4.8. So, on the basis of this shortest interval to have maximum predictive probability, we suggest that the double prior Gamma-Exponential distribution may be the best prior to be used for the unknown parameter λ of the Poisson distribution as compare to the single Gamma distribution as a prior. Also it is to be used especially for those cases in which the important prior information about λ is assumed to be coming from two or more different functions.

5. Real life example

Many of social, behavioral and other disciplines are replete with variables that are designed to count some real phenomena like number of children, number of arrests, number of traffic accidents on the high ways in 2008, number of marriages etc. These all are interesting characteristics of count variable say *X*, which is a Poisson random variable here. The data is taken from General Social Survey (GSS) of year 1996 conducted every two year by National Opinion Research Center of university of Chicago and available online at website http:// www.icpsr.umich.edu/GSS.

From the GSS96 data, the selected variable is “number of children” that’s (0,1,2,…,8 or above) and we explained it by another variable “marital status” of respondent like (married, divorced, widowed etc). We are interested in estimating the average number of children when the person is either married or divorced. Table 5.1 shows exact sample values of the data (missing observations are ignored).

From Table 5.1, we have two Poisson random variables X_1 and X_2. Here X_1 represents number of children of married couples and X_2 represents number of children of respondents in divorce case. Also total number of observations in both samples are $n_1 = 1383$ (married) and $n_2 = 455$ (divorced) respectively, with sums $\sum_{i=1}^{n_1} X_{1i} = 3024$ and $\sum_{i=1}^{n_2} X_{2i} = 455$.

5.1. Graphical comparison of priors with respect to posterior variances

The variances of the posterior distribution are calculated by assuming the different sets of values of hyperparameters, which are given in Tables 5.1.1 and 5.1.2 (see appendix).

It is observed from Tables 5.1.1 and 5.1.2, that the posterior variances calculated under double prior Gamma-Exponential distribution are reasonably less as compared to posterior variances computed under other priors. It is therefore preferred to use two or more informative priors for single unknown parameter instead of using single prior to get more precise Bayes estimates.

Fig. 1. Plot of posterior variances with $n_1 = 1383$.

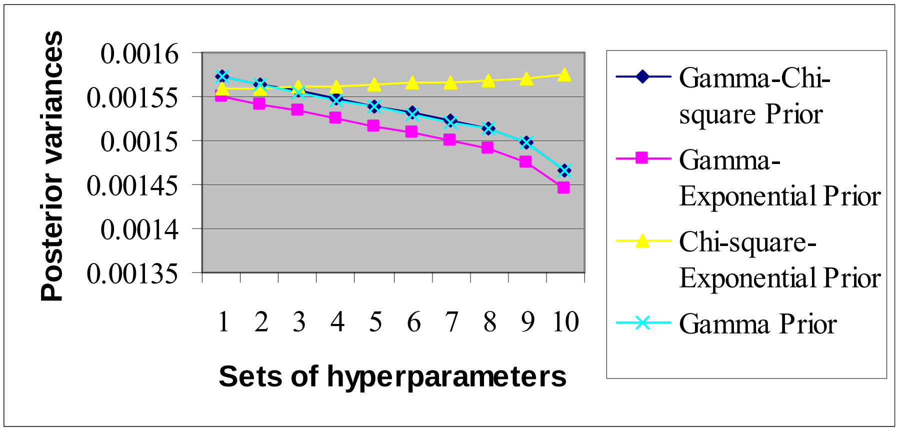
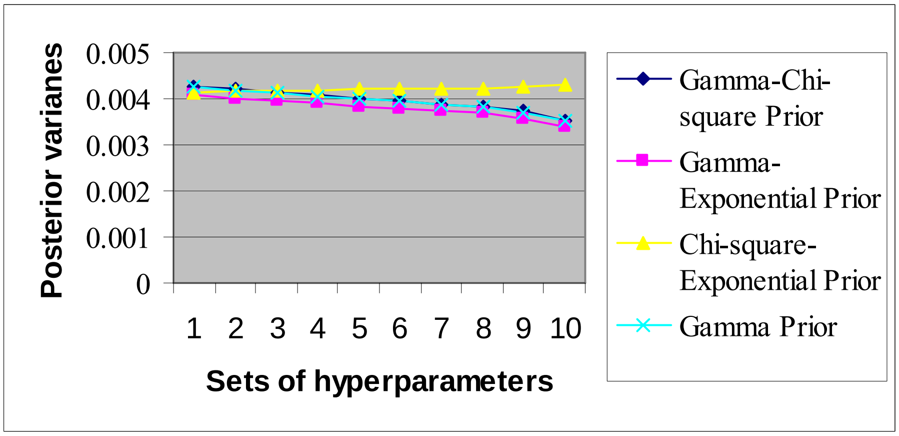


Fig. 2. Plot of posterior variances with $n_2 = 455$.



As it is clearly observed from figures 1 and 2 that posterior variances under the assumed priors get smaller and smaller as values of hyperparameters increase except under Chi-square-Exponential prior (under which posterior variances increase). The posterior variances for different values of hyperparameters under Gamma-Exponential prior are less as compared to the posterior variances under other priors, which ensures that under this prior we have more precise Bayes estimates.

5.2. Comparison of priors with respect to posterior predictive variances

The variances of the posterior predictive distribution are calculated by assuming different sets of values of hyperparameters are given in Tables 5.2.1 and 5.2.2 (see appendix):

It is clearly observed from Tables 5.2.1 and 5.2.2, that posterior predictive variances calculated under Gamma-Exponential double prior are more precise as compare to others, which means we can prefer this double prior for unknown parameter of Poisson distribution.

Fig. 3. Plot of posterior predictive variances with $n_1 = 1383$.

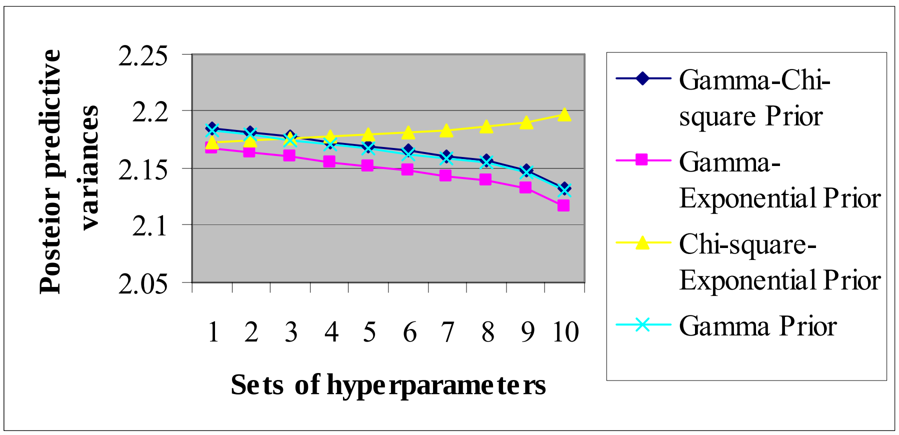
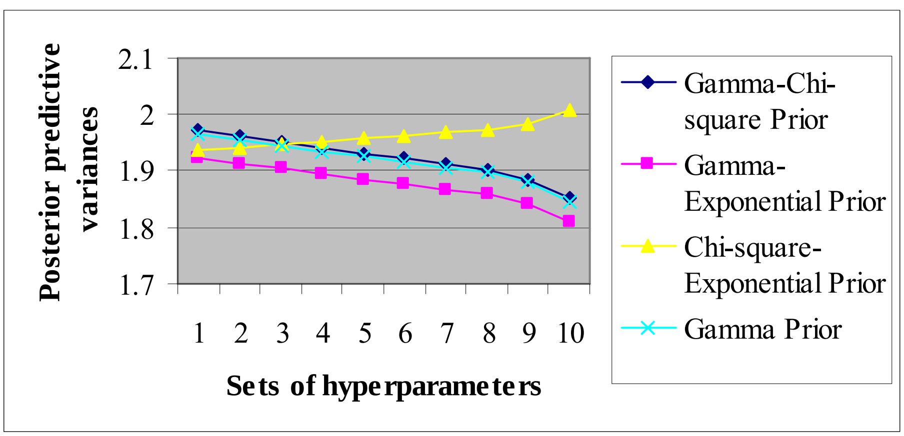


Fig. 4. Plot of posterior predictive variances with $n_2 = 455$.



In figures 3 and 4, posterior predictive variances are plotted against different sets of hyperparameters. It is observed that posterior predictive variances under Gamma-Exponential priors are less as compared to other priors. Also for first two sets of hyperparameters, the posterior predictive variances under Chi-square-Exponential prior are less than the variances under other two priors. It is therefore recommended to use double priors to incorporate prior information instead of assuming single prior for unknown parameter.

6. Conclusion

In this study we have provided the comparison of informative priors for the unknown parameter λ of the Poisson distribution. Under this comparison, four informative priors (Gamma -Chi-square prior, Gamma-Exponential prior, Chi-square-Exponential prior, Gamma prior) have been assumed and then their posterior distributions were derived, further posterior predictive distributions under these informative priors have also been developed. A comparison has been made on the basis of posterior variances, predictive posterior variance and posterior predictive probabilities for the fixed interval. In this comparison, we have observed that the double informative prior: Gamma-Exponential distribution possesses the less posterior and predictive posterior variances along with shortest intervals to have maximum posterior predictive probabilities, although the fixed interval is chosen in discrete distributions.

Also this new concept of using two or more informative priors for the single unknown parameter is useful in situations where we have different prior information available. In that case one can use two or more priors by incorporating that prior knowledge to get the precise estimates.

Finally we recommend on the basis of the above important characteristics that the Gamma-Exponential distribution double prior is the best choice for an informative prior for the unknown parameter of Poisson distribution.

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Appendix I

1. Poisson-Gamma distribution

A random variable X is said to possess a Poisson-Gamma distribution if its probability mass function has the following form

$$f(x\mid\alpha,\beta,v)=\frac{\beta^{\alpha}\Gamma(\alpha+x)v^{x}}{\Gamma(\alpha)x!(\beta+v)^{\alpha+x}},\quad \alpha,\beta,v>0,x=0,1,2,...$$

If X follows Poisson-Gamma distribution then it can be written as

$$X\sim Pg(\alpha,\beta,v).$$

(i). The mean of the said distribution is $E(X)=v\frac{\alpha}{\beta}$.

(ii). The variance of the distribution is $V(X)=\frac{\alpha v}{\beta}\left(1+\frac{v}{\beta}\right)$.

2. Tables of Sections 3.1, 4.5, 4.7 and 5.

Table 3.1.1
Variances of the posterior distribution using different priors with *n*=50

Hyper parameters		Gamma-Chi-square distribution				Gamma-Exponential distribution			
$a_i=b_i=c_i$		λ=2	λ=5	λ=8	λ=10	λ=2	λ=5	λ=8	λ=10
5	5	0.03458	0.08328	0.13197	0.16437	0.0292	0.07089	0.11238	0.14029
10	10	0.03118	0.07209	0.11314	0.14041	0.02244	0.05313	0.08363	0.1041
15	15	0.02832	0.06332	0.09822	0.12151	0.018	0.04142	0.06488	0.08046
20	20	0.02592	0.05615	0.08625	0.10641	0.0148	0.0333	0.05186	0.0642
25	25	0.02394	0.05025	0.07659	0.09413	0.0125	0.0275	0.04248	0.05252
30	30	0.02222	0.04542	0.06851	0.08388	0.01075	0.02314	0.03555	0.04376
35	35	0.02072	0.04125	0.06176	0.07545	0.00938	0.01979	0.0302	0.03715
40	40	0.01943	0.03775	0.05602	0.06824	0.00829	0.01715	0.02604	0.03195
50	50	0.01723	0.03209	0.04696	0.05683	0.00666	0.01333	0.02002	0.02444
70	70	0.01405	0.0244	0.03471	0.04159	0.00471	0.00887	0.01302	0.01579
		Chi-square-Exponential distribution				Gamma distribution			
$a_i=b_i=c_i$		λ=2	λ=5	λ=8	λ=10	λ=2	λ=5	λ=8	λ=10
5	5	0.03332	0.0819	0.13063	0.16318	0.03473	0.08425	0.13394	0.1671
10	10	0.02876	0.06964	0.11071	0.13797	0.03052	0.07228	0.11387	0.14164
15	15	0.02509	0.06003	0.09504	0.11815	0.0272	0.06274	0.09821	0.12184
20	20	0.02209	0.05235	0.08246	0.10261	0.02452	0.0551	0.08573	0.10625
25	25	0.01976	0.04602	0.07239	0.08984	0.02224	0.0489	0.07546	0.09335
30	30	0.01776	0.0409	0.06408	0.0795	0.02031	0.04374	0.06718	0.08282
35	35	0.01603	0.03658	0.05709	0.07074	0.01867	0.03947	0.06025	0.07405
40	40	0.01466	0.03299	0.05124	0.0635	0.01729	0.0358	0.05432	0.06663
50	50	0.01239	0.02723	0.04208	0.052	0.01501	0.03001	0.04503	0.05495
70	70	0.0093	0.01963	0.02995	0.03685	0.01181	0.02224	0.03262	0.0396

Table 3.1.2

Variances of the posterior distribution using different priors with *n*=80

Hyper parameters		Gamma-Chi-square distribution				Gamma-Exponential distribution			
$a_i = b_i = c_i$		λ=2	λ=5	λ=8	λ=10	λ=2	λ=5	λ=8	λ=10
5	5	0.02277	0.05556	0.08837	0.11028	0.02038	0.05	0.07959	0.09944
10	10	0.02127	0.05051	0.07985	0.09936	0.01701	0.04104	0.06502	0.08102
15	15	0.01991	0.04616	0.07255	0.09004	0.01445	0.0343	0.05416	0.06737
20	20	0.01869	0.04246	0.06618	0.08206	0.01251	0.02915	0.04582	0.05694
25	25	0.01766	0.0392	0.06077	0.07517	0.01094	0.02514	0.03933	0.04881
30	30	0.0167	0.03635	0.05601	0.06914	0.0097	0.02196	0.03417	0.04236
35	35	0.01584	0.03385	0.05182	0.06382	0.00867	0.01933	0.02999	0.03709
40	40	0.01509	0.03161	0.04815	0.05914	0.00781	0.0172	0.02655	0.03282
50	50	0.01373	0.02784	0.04193	0.05131	0.00648	0.01389	0.02128	0.02623
70	70	0.01165	0.02225	0.03284	0.03991	0.00475	0.00971	0.01466	0.01798
		Chi-square-Exponential distribution				**Gamma distribution**			
$a_i = b_i = c_i$		λ=2	λ=5	λ=8	λ=10	λ=2	λ=5	λ=8	λ=10
5	5	0.02222	0.05507	0.08787	0.10971	0.02284	0.05602	0.08924	0.11132
10	10	0.02015	0.04948	0.07876	0.09825	0.02098	0.05063	0.08024	0.09997
15	15	0.01836	0.04469	0.07099	0.08854	0.01938	0.046	0.07255	0.09032
20	20	0.01683	0.04058	0.06433	0.0802	0.018	0.04202	0.06602	0.08201
25	25	0.01549	0.03707	0.05861	0.07301	0.01681	0.03856	0.06031	0.0748
30	30	0.01433	0.03397	0.05364	0.06679	0.01572	0.03555	0.05536	0.06858
35	35	0.01331	0.03132	0.04933	0.06124	0.01474	0.03287	0.05104	0.06314
40	40	0.01239	0.02894	0.04547	0.05651	0.01388	0.03056	0.04719	0.05834
50	50	0.01087	0.02497	0.03907	0.04845	0.01241	0.02663	0.04083	0.0503
70	70	0.00861	0.0192	0.02981	0.03688	0.01023	0.02089	0.03156	0.03868

Table 3.1.3

Variances of the posterior distribution using different priors with *n*=100

Hyper Parameters		Gamma-Chi-square distribution				Gamma-Exponential distribution			
$a_i = b_i = c_i$		λ=2	λ=5	λ=8	λ=10	λ=2	λ=5	λ=8	λ=10
5	5	0.01854	0.04553	0.07248	0.09045	0.01696	0.04175	0.06649	0.08308
10	10	0.01752	0.0421	0.06662	0.08306	0.01458	0.03543	0.05626	0.07012
15	15	0.0166	0.0391	0.0616	0.07659	0.01272	0.03045	0.04824	0.06005
20	20	0.01579	0.03643	0.0571	0.07084	0.01123	0.02654	0.04187	0.05205
25	25	0.01502	0.03404	0.05314	0.06581	0.01	0.02334	0.03668	0.04553
30	30	0.01433	0.03196	0.04955	0.06128	0.00898	0.0207	0.03242	0.04024
35	35	0.0137	0.03003	0.04641	0.05728	0.00813	0.01852	0.02889	0.03579
40	40	0.01313	0.02831	0.04354	0.05364	0.00741	0.01667	0.02593	0.03211
50	50	0.01209	0.02534	0.03859	0.04741	0.00625	0.01375	0.02124	0.02625
70	70	0.01046	0.02077	0.0311	0.03798	0.00469	0.00989	0.0151	0.01858
		Chi-square-Exponential distribution				**Gamma distribution**			
$a_i = b_i = c_i$		λ=2	λ=5	λ=8	λ=10	λ=2	λ=5	λ=8	λ=10
5	5	0.01818	0.04514	0.07204	0.09003	0.0186	0.04581	0.07298	0.09117
10	10	0.0168	0.04138	0.06587	0.08226	0.01736	0.04212	0.06693	0.08345
15	15	0.01554	0.03805	0.0605	0.07551	0.01625	0.03895	0.06164	0.07673
20	20	0.01446	0.03514	0.05581	0.06957	0.01528	0.0361	0.0569	0.07084
25	25	0.0135	0.03253	0.05157	0.06427	0.01441	0.03361	0.05283	0.06559
30	30	0.01262	0.03023	0.04785	0.05955	0.01363	0.03136	0.04912	0.06096
35	35	0.01185	0.0282	0.04451	0.05542	0.01289	0.02937	0.04583	0.05679
40	40	0.01114	0.02635	0.04153	0.05166	0.01225	0.02754	0.04286	0.05307
50	50	0.00993	0.02318	0.03642	0.04528	0.01111	0.02445	0.03777	0.04666
70	70	0.00808	0.01842	0.02872	0.03561	0.00935	0.01973	0.03011	0.03704

Table 4.5.1
Posterior predictive variances using informative priors with *n*=50

Hyper Parameters		Gamma-Chi-square distribution				Gamma-Exponential distribution			
$a_i = b_i = c_i$		λ=2	λ=5	λ=8	λ=10	λ=2	λ=5	λ=8	λ=10
5	5	1.9546	4.70569	7.45952	9.28667	1.78043	4.32645	6.86571	8.55535
10	10	1.91766	4.43767	6.95634	8.63896	1.59392	3.76657	5.94564	7.38903
15	15	1.8827	4.20915	6.53412	8.07971	1.45771	3.35632	5.24751	6.51682
20	20	1.85545	4.01355	6.16881	7.60927	1.34725	3.03215	4.7204	5.84148
25	25	1.83049	3.84388	5.85992	7.19743	1.26141	2.77759	4.292	5.30106
30	30	1.81062	3.69945	5.57979	6.83631	1.19346	2.56736	3.94422	4.86318
35	35	1.79327	3.56802	5.34103	6.52421	1.13418	2.39641	3.65403	4.49547
40	40	1.77522	3.45176	5.12636	6.24491	1.08452	2.2483	3.41078	4.18656
50	50	1.74827	3.25505	4.76431	5.76808	1.00664	2.01399	3.02012	3.69168
70	70	1.70675	2.96049	4.21587	5.05029	0.89962	1.69425	2.48676	3.01523
		Chi-square-Exponential distribution				**Gamma distribution**			
$a_i = b_i = c_i$		λ=2	λ=5	λ=8	λ=10	λ=2	λ=5	λ=8	λ=10
5	5	1.8809	4.62723	7.38117	9.21918	1.94313	4.71726	7.50056	9.35037
10	10	1.76795	4.2835	6.81162	8.48318	1.86419	4.40601	6.95249	8.64004
15	15	1.66688	3.9919	6.31087	7.8697	1.79786	4.13915	6.48338	8.05085
20	20	1.58294	3.73757	5.89377	7.3358	1.73834	3.91411	6.08674	7.53371
25	25	1.5081	3.52392	5.5338	6.87857	1.6875	3.71572	5.74535	7.09428
30	30	1.44371	3.3327	5.21808	6.47347	1.64574	3.54408	5.44157	6.70467
35	35	1.38936	3.16481	4.93979	6.12178	1.60734	3.39002	5.1761	6.36233
40	40	1.34109	3.01764	4.69327	5.81179	1.57098	3.25795	4.94156	6.06526
50	50	1.25517	2.76137	4.27314	5.27655	1.51542	3.03128	4.5438	5.55707
70	70	1.12989	2.38405	3.64182	4.47818	1.42874	2.68999	3.94781	4.78716

Table 4.5.2
Posterior predictive variances using informative priors with *n*=80

Hyper Parameters		Gamma-Chi-square distribution				Gamma-Exponential distribution			
$a_i = b_i = c_i$		λ=2	λ=5	λ=8	λ=10	λ=2	λ=5	λ=8	λ=10
5	5	1.96916	4.81194	7.64584	9.54419	1.85214	4.54861	7.24706	9.04152
10	10	1.94395	4.62404	7.30895	9.08869	1.7173	4.1445	6.56373	8.18275
15	15	1.92246	4.46189	6.99924	8.68825	1.60327	3.80669	6.01063	7.48161
20	20	1.8991	4.31044	6.72658	8.32867	1.51386	3.52918	5.54604	6.89074
25	25	1.87803	4.17767	6.46697	8.00327	1.43367	3.29498	5.1583	6.39967
30	30	1.86379	4.05541	6.24793	7.7025	1.36635	3.09332	4.82024	5.97006
35	35	1.84808	3.94133	6.03523	7.43577	1.30862	2.91847	4.52823	5.60378
40	40	1.83103	3.84265	5.84683	7.18384	1.25911	2.76685	4.2757	5.28071
50	50	1.80852	3.66041	5.51408	6.7467	1.17298	2.51255	3.85482	4.74959
70	70	1.76617	3.37105	4.97639	6.04776	1.05017	2.14484	3.2429	3.97148
		Chi-square-Exponential distribution				**Gamma distribution**			
$a_i = b_i = c_i$		λ=2	λ=5	λ=8	λ=10	λ=2	λ=5	λ=8	λ=10
5	5	1.92446	4.76095	7.60394	9.49999	1.96219	4.82263	7.67921	9.58563
10	10	1.84152	4.52789	7.20052	8.99354	1.9093	4.60515	7.3018	9.10149
15	15	1.77375	4.30833	6.85269	8.54483	1.86302	4.41566	6.96904	8.6685
20	20	1.70803	4.11832	6.53023	8.13758	1.81799	4.24351	6.66292	8.27979
25	25	1.65067	3.94382	6.24175	7.77497	1.77914	4.08725	6.39287	7.93193
30	30	1.59807	3.78875	5.98225	7.44062	1.7425	3.94193	6.14751	7.61138
35	35	1.55064	3.64466	5.74495	7.13995	1.71066	3.81508	5.92417	7.32306
40	40	1.50726	3.51524	5.52243	6.86082	1.68046	3.69776	5.70929	7.06003
50	50	1.42574	3.27966	5.13425	6.37088	1.62642	3.48922	5.34683	6.59083
70	70	1.30383	2.91027	4.51364	5.58223	1.54253	3.15263	4.76518	5.84108

Table 4.5.3

Posterior predictive variances using informative priors with *n*=100

Hyper Parameters		Gamma-Chi-square distribution				Gamma-Exponential distribution			
$a_i = b_i = c_i$		λ=2	λ=5	λ=8	λ=10	λ=2	λ=5	λ=8	λ=10
5	5	1.97641	4.84632	7.715	9.63117	1.88001	4.631	7.3839	9.21372
10	10	1.95351	4.69583	7.42958	9.25496	1.76464	4.28642	6.80859	8.48964
15	15	1.9327	4.55172	7.17705	8.91825	1.66648	3.99031	6.31973	7.87146
20	20	1.91583	4.42856	6.93504	8.6056	1.58236	3.73897	5.89944	7.33521
25	25	1.89989	4.31151	6.71605	8.32244	1.51048	3.52445	5.53537	6.88044
30	30	1.88245	4.20101	6.51532	8.0583	1.44573	3.33261	5.2183	6.47883
35	35	1.86843	4.09899	6.3312	7.81714	1.38985	3.16596	4.94064	6.12236
40	40	1.85515	4.00767	6.16002	7.58799	1.3403	3.01534	4.6883	5.81124
50	50	1.83205	3.83889	5.84871	7.18385	1.2567	2.76108	4.2688	5.27733
70	70	1.79369	3.56274	5.33116	6.51378	1.12977	2.38216	3.64124	4.47684
		Chi-square-Exponential distribution				Gamma distribution			
$a_i = b_i = c_i$		λ=2	λ=5	λ=8	λ=10	λ=2	λ=5	λ=8	λ=10
5	5	1.93563	4.80999	7.67937	9.58849	1.96946	4.85529	7.74095	9.66384
10	10	1.87235	4.61363	7.35538	9.17444	1.92526	4.67733	7.42942	9.26462
15	15	1.81131	4.43529	7.04702	8.7949	1.88648	4.51771	7.14822	8.9019
20	20	1.75815	4.26701	6.77744	8.45282	1.84849	4.37013	6.88642	8.56981
25	25	1.70689	4.11605	6.52347	8.12902	1.81505	4.23093	6.65114	8.26353
30	30	1.65997	3.97735	6.29299	7.83548	1.78341	4.10827	6.43147	7.98386
35	35	1.61811	3.84796	6.07529	7.56216	1.75223	3.99282	6.23111	7.72442
40	40	1.57799	3.72328	5.87907	7.30971	1.72481	3.88384	6.0448	7.4858
50	50	1.50506	3.51062	5.5172	6.85555	1.677	3.69204	5.70838	7.04651
70	70	1.38552	3.1572	4.92906	6.10473	1.5978	3.37459	5.14981	6.32995

Table 4.7.1

Posterior predictive probabilities with λ=2 and *n*=100

Names of prior	Hyper parameters	Posterior parameters	Predictive probabilities for fixed interval [0,5]
Gamma-Chi-square distribution	$a_1 = b_1 = c_1 = 10$	$\alpha_1 = 214.0884,\ \beta_1 = 110.5$	0.98513
	$a_1 = b_1 = c_1 = 40$	$\alpha_1 = 259.0884,\ \beta_1 = 140.5$	0.98811
	$a_1 = b_1 = c_1 = 70$	$\alpha_1 = 304.0884,\ \beta_1 = 170.5$	0.98981
	$a_1 = b_1 = c_1 = 100$	$\alpha_1 = 349.0884,\ \beta_1 = 200.5$	0.99089
Gamma-Exponential distribution	$a_2 = b_2 = c_2 = 10$	$\alpha_2 = 210.0884,\ \beta_2 = 120.0$	0.99053
	$a_2 = b_2 = c_2 = 40$	$\alpha_2 = 240.0884,\ \beta_2 = 180.0$	0.99738
	$a_2 = b_2 = c_2 = 70$	$\alpha_2 = 270.0884,\ \beta_2 = 240.0$	0.99887
	$a_2 = b_2 = c_2 = 100$	$\alpha_2 = 300.0884,\ \beta_2 = 300.0$	**0.99938**
Chi-square - Exponential distribution	$a_3 = c_3 = 10$	$\alpha_3 = 205.0884,\ \beta_3 = 110.5$	0.98768
	$a_3 = c_3 = 40$	$\alpha_3 = 220.0884,\ \beta_3 = 140.5$	0.99433
	$a_3 = c_3 = 70$	$\alpha_3 = 235.0884,\ \beta_3 = 170.5$	0.99691
	$a_3 = c_3 = 100$	$\alpha_3 = 250.0884,\ \beta_3 = 200.5$	0.99811
Gamma distribution	$a_4 = c_4 = 10$	$\alpha_4 = 210.0884,\ \beta_4 = 110.0$	0.98603
	$a_4 = c_4 = 40$	$\alpha_4 = 240.0884,\ \beta_4 = 140.0$	0.99141

	$a_4 = c_4 = 70$	$\alpha_4 = 270.0884,\ \beta_4 = 170.0$	0.99398
	$a_4 = c_4 = 100$	$\alpha_4 = 300.0884,\ \beta_4 = 200.0$	0.99541

Table 4.7.2
Posterior predictive probabilities with λ=10 and n=100

Names of prior	Hyper parameters	Posterior parameters	Predictive probabilities for fixed interval [0,10]
Gamma-Chi-square distribution	$a_1 = b_1 = c_1 = 10$	$\alpha_1 = 1014.298,\ \beta_1 = 110.5$	0.684099
	$a_1 = b_1 = c_1 = 40$	$\alpha_1 = 1059.298,\ \beta_1 = 140.5$	0.858069
	$a_1 = b_1 = c_1 = 70$	$\alpha_1 = 1104.298,\ \beta_1 = 170.5$	0.933878
	$a_1 = b_1 = c_1 = 100$	$\alpha_1 = 1149.298,\ \beta_1 = 200.5$	0.967114
Gamma-Exponential distribution	$a_2 = b_2 = c_2 = 10$	$\alpha_2 = 1010.298,\ \beta_2 = 120.0$	0.771511
	$a_2 = b_2 = c_2 = 40$	$\alpha_2 = 1040.298,\ \beta_2 = 180.0$	0.965417
	$a_2 = b_2 = c_2 = 70$	$\alpha_2 = 1070.298,\ \beta_2 = 240.0$	0.993628
	$a_2 = b_2 = c_2 = 100$	$\alpha_2 = 1100.298,\ \beta_2 = 300.0$	**0.998498**
Chi-square - Exponential distribution	$a_3 = c_3 = 10$	$\alpha_3 = 1005.298,\ \beta_3 = 110.5$	0.693855
	$a_3 = c_3 = 40$	$\alpha_3 = 1020.298,\ \beta_3 = 140.5$	0.881086
	$a_3 = c_3 = 70$	$\alpha_3 = 1035.298,\ \beta_3 = 170.5$	0.953830
	$a_3 = c_3 = 100$	$\alpha_3 = 1050.298,\ \beta_3 = 200.5$	0.981172
Gamma distribution	$a_4 = c_4 = 10$	$\alpha_4 = 1010.298,\ \beta_4 = 110.0$	0.683453
	$a_4 = c_4 = 40$	$\alpha_4 = 1040.298,\ \beta_4 = 140.0$	0.867351
	$a_4 = c_4 = 70$	$\alpha_4 = 1070.298,\ \beta_4 = 170.0$	0.943385
	$a_4 = c_4 = 100$	$\alpha_4 = 1100.298,\ \beta_4 = 200.0$	0.974385

Table 5.1

Count variable	**Marital Status**	
Number of children	Married	Divorced
X	Frequency	Frequency
0	205	76
1	216	101
2	473	141
3	265	76
4	134	37
5	44	12
6	21	7
7	15	4
8 or more	10	1
Total	1383	455

Table 5.1.1

Variances of the posterior distribution with $n_1 = 1383$

Hyper parameters $a_i = b_i = c_i$		Gamma-Chi-square distribution	Gamma-Exponential distribution	Chi-square-Exponential distribution	Gamma distribution
5	5	0.001573	0.00155	0.001559	0.001572
10	10	0.001564	0.001541	0.00156	0.001564
15	15	0.001556	0.001533	0.001561	0.001555
20	20	0.001547	0.001525	0.001562	0.001546
25	25	0.001539	0.001516	0.001564	0.001538
30	30	0.001531	0.001508	0.001565	0.00153
35	35	0.001522	0.0015	0.001566	0.001521
40	40	0.001514	0.001492	0.001568	0.001513
50	50	0.001498	0.001476	0.00157	0.001497
70	70	0.001466	0.001446	0.001575	0.001466

Table 5.1.2

Variances of the posterior distribution with $n_2 = 455$

Hyper parameters $a_i = b_i = c_i$		Gamma-Chi-square distribution	Gamma-Exponential distribution	Chi-square-Exponential distribution	Gamma distribution
5	5	0.004272	0.004083	0.004151	0.004263
10	10	0.004204	0.00402	0.004163	0.004195
15	15	0.004138	0.003958	0.004174	0.004129
20	20	0.004073	0.003898	0.004186	0.004064
25	25	0.004011	0.00384	0.004197	0.004002
30	30	0.00395	0.003783	0.004209	0.003941
35	35	0.00389	0.003728	0.00422	0.003882
40	40	0.003833	0.003674	0.004232	0.003824
50	50	0.003722	0.003571	0.004255	0.003713
70	70	0.003516	0.003378	0.004301	0.003508

Table 5.2.1

Variances of the posterior predictive distribution with $n_1 = 1383$

Hyper parameters $a_i = b_i = c_i$		Gamma-Chi-square distribution	Gamma-Exponential distribution	Chi-square-Exponential distribution	Gamma distribution
5	5	2.185945	2.168216	2.173428	2.183849
10	10	2.181686	2.16405	2.175223	2.179597
15	15	2.177459	2.159914	2.177019	2.175375
20	20	2.173261	2.155806	2.178814	2.171183
25	25	2.169093	2.151728	2.180609	2.167021
30	30	2.164955	2.147678	2.182405	2.162888
35	35	2.160845	2.143657	2.1842	2.158785
40	40	2.156765	2.139664	2.185995	2.154711
50	50	2.14869	2.13176	2.189586	2.146647
70	70	2.132873	2.116278	2.196767	2.130853

Table 5.2.2

Variances of the posterior predictive distribution with $n_2 = 455$

Hyper parameters $a_i = b_i = c_i$		Gamma-Chi-square distribution	Gamma-Exponential distribution	Chi-square-Exponential distribution	Gamma distribution
5	5	1.971699	1.923232	1.936482	1.965132
10	10	1.96124	1.913494	1.941864	1.954732
15	15	1.951003	1.903958	1.947246	1.944554
20	20	1.940982	1.89462	1.952628	1.934591
25	25	1.93117	1.885473	1.95801	1.924835
30	30	1.92156	1.876511	1.963392	1.915281
35	35	1.912147	1.867728	1.968775	1.905923
40	40	1.902925	1.85912	1.974157	1.896753
50	50	1.885027	1.842406	1.984921	1.878961
70	70	1.85128	1.810855	2.006449	1.845413