Characterizations of the Moments of the Purely Diagonal Bilinear Time Series Model of Order One

By

Ohakwe, Johnson\(^{(1)}\) and Iheanyi S. Iwueze\(^{(2)}\)

1. Department of Statistics, Faculty of Biological and Physical Sciences, Abia State University, P.M.B. 2000, Uturu, Abia State, Nigeria.

2. Department of Physical Sciences, College of Natural and Applied Sciences, Renaissance University, P.M.B 01183, Ugbawka, Enugu State, Nigeria.

Abstract: In this paper, we study the similarities and dissimilarities between a purely diagonal bilinear process of order one [PDB(1)] and a moving average process of order one [MA(1)] by comparing their first, second and fourth order moments. The well known similarity between their covariance structure was discovered to be true only when \(0 < \rho_1 < 0.16\). With respect to the fourth moments, the PDB(1) process identifies as an autoregressive moving average process of order \(p = 1\) and \(q = 1\) while the equivalent non-zero mean MA(1) process identifies as an MA(1) process.

Key Words: Purely bilinear process, Moving average process, Moments, Autocorrelation function, Covariance structure.

INTRODUCTION

It has been established that linear and bilinear processes have similar covariance structure (Granger and Anderson ,1978, Subba Rao, 1981, Tong, 1981, Bhaskara Rao et al., 1983 and Akamanam et al., 1986). By similarity in covariance structure, we mean that the two models have significant autocorrelations at the same lag(s). but within these similarities in covariance structure are hidden dissimilarities. The problem of differentiating a linear autoregressive moving average (ARMA) process from a bilinear process has therefore engaged the attention of many authors. Third order moments and cumulants are the widely accepted method of differentiating
between the two competing models (Gabr, 1981), Sessay and Suba Rao, 1991, Oyet and Iwueze, 1993, and Iwueze and Chikezie, 2006). No attention has been paid to the inherent differences between linear and bilinear processes under covariance analysis. Also, little attention has been paid to the fourth moments or the time series properties of the squares of linear and bilinear process.

Let \( e_t, t \in \mathbb{Z} \) be a sequence of independently identically distributed random variables with \( E(e_t) = 0 \) and \( E(e_t^2) = \sigma_t^2 < \infty \). Let \( a_1, a_2, ..., a_r, b_1, b_2, ..., b_h, \ \theta_{ij}, 1 \leq i \leq m, 1 \leq j \leq l \) be real constants. The general bilinear autoregressive moving average process of order \( (r, h, m, l) \), denoted by \( \text{BARMA}(r, h, m, l) \), is defined by

\[
X_t = \sum_{i=1}^{r} a_i X_{t-i} + \sum_{j=1}^{h} b_j e_{t-j} + \sum_{i=1}^{m} \sum_{j=1}^{l} \theta_{ij} X_{t-i} e_{t-j} + e_t
\]  

(1.1)

The model (1.1) is said to be a purely or completely bilinear process when \( a_i = 0, \forall i = 1, 2, ..., m; b_j = 0, \forall j = 1, 2, ..., h \). A purely bilinear process is said to be a purely diagonal bilinear process if in addition \( \theta_{ij} = 0, \forall i \neq j \). For the purely diagonal process of order \( q \), denoted by \( \text{PDB}(q) \), model (1.1) becomes

\[
X_t = \sum_{j=1}^{q} \theta_j X_{t-j} e_{t-j} + e_t
\]  

(1.2)

The stationarity and invertibility conditions of the \( \text{PDB}(q) \) process have been established by Guegan and Pham (1987). Okoye (1992) established that the covariance function of (1.2) is the same as that of a non-zero mean moving average process of order \( q \), denoted by \( \text{MA}(q) \), and given by

\[
X_t = \beta_0 + \sum_{j=1}^{q} \beta_j u_{t-j} + u_t
\]  

(1.3)
where $u_t, t \in Z$ with $E(u_t) = 0$ and $E(u_t^2) = \sigma_t^2 < \infty$ may not be a purely random process.

The purpose of this paper is to carry out an in-depth analysis of the first, second and fourth order moments of the purely diagonal process of order one [PDB(1)] and that of the competing linear moving average process of order one [MA(1)] with a view to providing an alternative differentiating technique.

**DISSIMILARITIES BETWEEN PDB(1) AND MA(1) PROCESSES UNDER COVARIANCE ANALYSIS**

$X_t, t \in Z$ is a purely diagonal bilinear time series model of order one, denoted by PDB(1), if

$$X_t = \theta_1 X_{t-1} e_{t-1} + e_t$$

(2.1)

where $e_t, t \in Z$ is a sequence of independent identically distributed random process with $E(e_t) = 0$ and $E(e_t^2) = \sigma_t^2$. The linear equivalent of model (2.1) is a non-zero mean moving average process of order one, denoted by MA(1), and is given by

$$X_t = \beta_0 + \beta_1 u_{t-1} + u_t$$

(2.2)

where $u_t, t \in Z$ with $E(u_t) = 0$ and $E(u_t^2) = \sigma_t^2 < \infty$ may not be a random process.

Given model (2.1), the following results were established with

$\lambda_1 = \sigma_1 \theta_1$ and $|\lambda_1| < 1$. 

$$E(X_t) = \sigma_t^2 \theta_1 = \sigma_t \lambda_1$$

(2.3)
\[
R(k) = \begin{cases} 
\frac{\sigma_i^2 (1 + \sigma_i^2 \theta_i^2 + \sigma_i^4 \theta_i^4)}{1 - \sigma_i^2 \theta_i^2} = \frac{\sigma_i^2 (1 + \lambda_i^2 + \lambda_i^4)}{1 - \lambda_i^2}, & k = 0 \\
\sigma_i^2 \theta_i^2 = \sigma_i^2 \lambda_i^2, & k = 1 \\
0, & k \geq 2
\end{cases}
\]

and
\[
\rho_k = \begin{cases} 
1, & k = 0 \\
\frac{\sigma_i^2 \theta_i^2 (1 - \sigma_i^2 \theta_i^2)}{1 + \sigma_i^2 \theta_i^2 + \sigma_i^4 \theta_i^4} = \frac{\lambda_i^2 (1 - \lambda_i^2)}{1 + \lambda_i^2 + \lambda_i^4}, & k = 1 \\
0, & k \geq 2
\end{cases}
\]

Furthermore, given model (2.2), the following are easily established (Box et al., 1994)
\[
E(X_i) = \beta_0 
\]

\[
R(k) = \begin{cases} 
\sigma_i^2 (1 + \beta_i^2), & k = 0 \\
\sigma_i^2 \beta_i, & k = 1 \\
0, & k \geq 2
\end{cases}
\]

and
\[
\rho_k = \begin{cases} 
1, & k = 0 \\
\frac{\beta_i}{1 + \beta_i^2}, & k = 1 \\
0, & k \geq 2
\end{cases}
\]

with \(|\beta_i| < 1\) for invertibility.

In obtaining the maximum value of \(\rho_1\) for model (2.1), PDB(1) process, we differentiate (2.5) with respect to \(\lambda_i\), equating the result to zero and solving the resultant equation for \(\lambda_i\) to obtain
\[
1 - 2 \lambda_i^2 - 2 \lambda_i^4 = 0
\]
Solving (2.9) for feasible values of $\lambda_1$, we obtain $\lambda_1 = \sqrt{0.3660} = \pm 0.6051$.

Similarly, the maximum value of $\rho_1$ for model (2.2), MA(1) process, is at $\beta_1 = \pm 1$ which when substituted into (2.8) yields $|\rho_1| < 0.5$ (Chatfield, 2004).

Table 1 shows the computed $\rho_1$ values for the PDB(1) and MA(1) processes for given $\lambda_1$ and $\beta_1$ values, respectively. Furthermore, Figure 1 shows the plot of $\rho_1$ against all admissible values of $\lambda_1$ for the PDB(1) process while Figure 2 shows the plot of $\rho_1$ against all admissible values of $\beta_1$ for the MA(1) process. For want of space, Table 1 is an abridged table; the full table has $\lambda_1 = [-0.99, -0.98, \ldots, 0.98, 0.99]$ and $\beta_1 = [-0.99, -0.98, \ldots, 0.98, 0.99]$. The plots of Figure 1 and Figure 2 were based on the full table which is available in Ohakwe (2008).

Table 1: First order autocorrelation for the PDB(1) and MA(1) processes.

<table>
<thead>
<tr>
<th>S/NO.</th>
<th>PDB(1) process</th>
<th>MA(1) process</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\beta_1$</td>
<td>$\rho_1$</td>
</tr>
<tr>
<td>1</td>
<td>-0.9</td>
<td>0.0624</td>
</tr>
<tr>
<td>2</td>
<td>-0.8</td>
<td>0.1124</td>
</tr>
<tr>
<td>3</td>
<td>-0.7</td>
<td>0.1444</td>
</tr>
<tr>
<td>4</td>
<td>-0.6</td>
<td>0.1547</td>
</tr>
<tr>
<td>5</td>
<td>-0.5</td>
<td>0.1429</td>
</tr>
<tr>
<td>6</td>
<td>-0.4</td>
<td>0.1134</td>
</tr>
<tr>
<td>7</td>
<td>-0.3</td>
<td>0.0746</td>
</tr>
<tr>
<td>8</td>
<td>-0.2</td>
<td>0.0369</td>
</tr>
<tr>
<td>9</td>
<td>-0.1</td>
<td>0.0098</td>
</tr>
<tr>
<td>10</td>
<td>0.0</td>
<td>0.0000</td>
</tr>
<tr>
<td>11</td>
<td>0.1</td>
<td>0.0098</td>
</tr>
<tr>
<td>12</td>
<td>0.2</td>
<td>0.0369</td>
</tr>
<tr>
<td>13</td>
<td>0.3</td>
<td>0.0746</td>
</tr>
<tr>
<td>14</td>
<td>0.4</td>
<td>0.1134</td>
</tr>
<tr>
<td>15</td>
<td>0.5</td>
<td>0.1429</td>
</tr>
<tr>
<td>16</td>
<td>0.6</td>
<td>0.1547</td>
</tr>
<tr>
<td>17</td>
<td>0.7</td>
<td>0.1444</td>
</tr>
<tr>
<td>18</td>
<td>0.8</td>
<td>0.1124</td>
</tr>
<tr>
<td>19</td>
<td>0.9</td>
<td>0.0624</td>
</tr>
</tbody>
</table>

From Table 1, Figure 1 and Figure 2, the following dissimilarities are evident:
1. **Intervals for** $\rho_1$: For the PDB(1) process, $0 < \rho_1 < 0.16$ while for the MA(1) process, $|\rho_1| < 0.5$. This means that only MA(1) processes with $0 < \rho_1 < 0.16$ can be compared with PDB(1) processes.

![Figure 1](image1.png)

**Figure 1:** Plot of $\rho_1$ against $\lambda_1$ for the PDB(1) process.

![Figure 2](image2.png)

**Figure 2:** Plot of $\rho_1$ against $\beta_1$ for the MA(1) process.
2. Graphs of $\rho_1$ against $\lambda_1$ or $\beta_1$: For the PDB(1) process, the graph of $\rho_1$ against $\lambda_1$ is wave-like with three turning points (-0.6051, 0.1547), (0, 0), and (0.6051, 0.1547) while for the MA(1) process, the graph of $\rho_1$ against $\beta_1$ looks like an arctangent.

RELATIONSHIP BETWEEN THE PARAMETERS OF THE PDB(1) PROCESS AND ITS EQUIVALENT MA(1) PROCESSES

Considering that the PDB(1) process identifies as a non-zero mean MA(1) process, we establish the relationship between their parameters by computing the values of $\beta_0, \beta_1$ and $\sigma^2_1$ for given $\theta_1$ and $\sigma^2_1$ values. To do this we used the method of moments estimation procedure which involves equating the first and second moments of the PDB(1) process to their counterparts of the non-zero mean MA(1) process. That is:

Equating Means: $\sigma^2_1 \theta_1 = \sigma_1 \lambda_1 = \beta_0$  \hspace{1cm} (3.1)

Equating Variances:

$$\frac{\sigma^2_1 \left(1 + \sigma^2_1 \theta_1^2 + \sigma^4_1 \theta_1^4\right)}{1 - \sigma^2_1 \theta_1^2} = \frac{\sigma^2_1 \left(1 + \lambda_1^2 + \lambda_1^4\right)}{1 - \lambda_1^2} = \sigma^2_2 \left(1 + \beta_1^2\right)$$  \hspace{1cm} (3.2)

Equating first order autocorrelations:

$$\frac{\sigma^2_1 \theta_1^2 \left(1 - \sigma^2_1 \theta_1^2\right)}{1 + \sigma^2_1 \theta_1^2 + \sigma^4_1 \theta_1^4} = \frac{\lambda_1^2 \left(1 - \lambda_1^2\right)}{1 + \lambda_1^2 + \lambda_1^4} = \frac{\beta_1}{1 + \beta_1^2}$$  \hspace{1cm} (3.3)

Computation of $\beta_0$ from (3.1) is easy but the computation of $\beta_1$ from (3.3) requires solving the quadratic equation

$$\lambda_1^2 \left(1 - \lambda_1^2\right) \beta_1^2 - \left(1 + \lambda_1^2 + \lambda_1^4\right) \beta_1 + \lambda_1^2 \left(1 - \lambda_1^2\right) = 0$$  \hspace{1cm} (3.4)

whose solution is
\[
\beta_1 = \frac{(1 + \lambda_1^2 + \lambda_1^4) \pm \sqrt{(1 + \lambda_1^2 + \lambda_1^4)^2 - 4 \lambda_1^4 (1 - \lambda_1^2)^2}}{2 \lambda_1^2 (1 - \lambda_1^2)}
\]  \hspace{1cm} (3.5)

Simulation concerning (3.5) reveals that for all \(|\lambda_1| < 1\)

\[
\frac{(1 + \lambda_1^2 + \lambda_1^4) + \sqrt{(1 + \lambda_1^2 + \lambda_1^4)^2 - 4 \lambda_1^4 (1 - \lambda_1^2)^2}}{2 \lambda_1^2 (1 - \lambda_1^2)} > 6
\]  \hspace{1cm} (3.6)

and

\[
\frac{(1 + \lambda_1^2 + \lambda_1^4) - \sqrt{(1 + \lambda_1^2 + \lambda_1^4)^2 - 4 \lambda_1^4 (1 - \lambda_1^2)^2}}{2 \lambda_1^2 (1 - \lambda_1^2)} < 1
\]  \hspace{1cm} (3.7)

Considering the invertibility condition, \(|\beta_1| < 1\), it is clear that the feasible solution of \(\beta_1\) is

\[
\beta_1 = \frac{(1 + \lambda_1^2 + \lambda_1^4) - \sqrt{(1 + \lambda_1^2 + \lambda_1^4)^2 - 4 \lambda_1^4 (1 - \lambda_1^2)^2}}{2 \lambda_1^2 (1 - \lambda_1^2)}
\]  \hspace{1cm} (3.8)

Finally, \(\sigma_2^2\) can easily be obtained from (3.2) as

\[
\sigma_2^2 = \frac{\sigma_1^2 (1 + \sigma_1^2 \theta_1^2 + \sigma_1^4 \theta_1^4)}{(1 + \beta_1^2) (1 - \sigma_1^2 \theta_1^2)} = \frac{\sigma_1^2 (1 + \lambda_1^2 + \lambda_1^4)}{(1 + \beta_1^2) (1 - \lambda_1^2)}
\]  \hspace{1cm} (3.9)

It is clear from (2.3), (2.4), (3.1) and (3.9) that \(E(X, k), R(k), \beta_0\) and \(\sigma_2^2\) have different values for fixed \(\lambda_1 = \sigma_1 \theta_1\) due to the variations in \(\sigma_1\) and \(\theta_1\). On the other hand, (2.5) and (3.8) show that for fixed \(\lambda_1 = \sigma_1 \theta_1\), \(\rho_1\) and \(\beta_1\) are constants independent of the variations in \(\sigma_1\) and \(\theta_1\). The computations of \(\rho_1\) and \(\beta_1\) for fixed \(\lambda_1 = \sigma_1 \theta_1\) are given in Table 2.

**RELATIONSHIP BETWEEN THE PARAMETERS OF AN MA(1) PROCESS AND ITS EQUIVALENT PDB(1) PROCESSES**

Considering that an MA(1) process may also be wrongly identified as a
PDB(1) process, we establish the relationship between their parameters by computing
the values of $\lambda_i$ and $\sigma_i^2$ for given values of $\beta_0$, $\beta_1$ and $\sigma_i^2$. We would also use the
method of moments as described in (3.2) and (3.3). Solving for $\lambda_i^2$ using (3.3), we obtain

$$\lambda_i^2 = \left(1 - \beta_1 + \beta_i^2\right) \pm \sqrt{\left(1 - \beta_1 + \beta_i^2\right)^2 - 4\beta_1\left(1 + \beta_1 + \beta_i^2\right)}$$

(4.1)

Solving for $\sigma_i^2$ using (3.2), we obtain

$$\sigma_i^2 = \frac{\sigma_i^2\left(1 + \beta_i^2\right)\left(1 - \sigma_i^2\theta_i^2\right)}{1 + \sigma_i^2\theta_i^2 + \sigma_i^4\theta_i^4} = \frac{\sigma_i^2\left(1 + \beta_i^2\right)\left(1 - \lambda_i^2\right)}{1 + \lambda_i^2 + \lambda_i^4}$$

(4.2)

Having observed that $|\rho_1| < 0.5$ for an MA(1) process and $0 < \rho_1 < 0.16$
for a PDB(1) process, we shall therefore consider the region where one model can

Table 2: Relationship between the parameters of a PDB(1) process and its
equivalent MA(1) processes

<table>
<thead>
<tr>
<th>S/NO.</th>
<th>PDB(1) process</th>
<th>Equivalent MA(1) process</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\beta_1$</td>
<td>$\rho_1$</td>
</tr>
<tr>
<td>1</td>
<td>-0.9</td>
<td>0.0624</td>
</tr>
<tr>
<td>2</td>
<td>-0.8</td>
<td>0.1124</td>
</tr>
<tr>
<td>3</td>
<td>-0.7</td>
<td>0.1444</td>
</tr>
<tr>
<td>4</td>
<td>-0.6</td>
<td>0.1547</td>
</tr>
<tr>
<td>5</td>
<td>-0.5</td>
<td>0.1429</td>
</tr>
<tr>
<td>6</td>
<td>-0.4</td>
<td>0.1134</td>
</tr>
<tr>
<td>7</td>
<td>-0.3</td>
<td>0.0746</td>
</tr>
<tr>
<td>8</td>
<td>-0.2</td>
<td>0.0369</td>
</tr>
<tr>
<td>9</td>
<td>-0.1</td>
<td>0.0098</td>
</tr>
<tr>
<td>10</td>
<td>0.0</td>
<td>0.0000</td>
</tr>
<tr>
<td>11</td>
<td>0.1</td>
<td>0.0098</td>
</tr>
<tr>
<td>12</td>
<td>0.2</td>
<td>0.0369</td>
</tr>
<tr>
<td>13</td>
<td>0.3</td>
<td>0.0746</td>
</tr>
<tr>
<td>14</td>
<td>0.4</td>
<td>0.1134</td>
</tr>
<tr>
<td>15</td>
<td>0.5</td>
<td>0.1429</td>
</tr>
<tr>
<td>16</td>
<td>0.6</td>
<td>0.1547</td>
</tr>
<tr>
<td>17</td>
<td>0.7</td>
<td>0.1444</td>
</tr>
<tr>
<td>18</td>
<td>0.8</td>
<td>0.1124</td>
</tr>
<tr>
<td>19</td>
<td>0.9</td>
<td>0.0624</td>
</tr>
</tbody>
</table>
Table 3: Relationship between the parameters of an MA(1) process and its equivalent PDB(1) processes

<table>
<thead>
<tr>
<th>S/NO</th>
<th>MA(1) Process</th>
<th>Equivalent PDB(1) process</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\beta_1$</td>
<td>$\lambda_1^2$ (+ sign)</td>
</tr>
<tr>
<td>1</td>
<td>0.01</td>
<td>0.9700</td>
</tr>
<tr>
<td>2</td>
<td>0.02</td>
<td>0.9399</td>
</tr>
<tr>
<td>3</td>
<td>0.03</td>
<td>0.9098</td>
</tr>
<tr>
<td>4</td>
<td>0.04</td>
<td>0.8795</td>
</tr>
<tr>
<td>5</td>
<td>0.05</td>
<td>0.8490</td>
</tr>
<tr>
<td>6</td>
<td>0.06</td>
<td>0.8182</td>
</tr>
<tr>
<td>7</td>
<td>0.07</td>
<td>0.7870</td>
</tr>
<tr>
<td>8</td>
<td>0.08</td>
<td>0.7552</td>
</tr>
<tr>
<td>9</td>
<td>0.09</td>
<td>0.7227</td>
</tr>
<tr>
<td>10</td>
<td>0.10</td>
<td>0.6891</td>
</tr>
<tr>
<td>11</td>
<td>0.11</td>
<td>0.6541</td>
</tr>
<tr>
<td>12</td>
<td>0.12</td>
<td>0.6170</td>
</tr>
<tr>
<td>13</td>
<td>0.13</td>
<td>0.5768</td>
</tr>
<tr>
<td>14</td>
<td>0.14</td>
<td>0.5313</td>
</tr>
<tr>
<td>15</td>
<td>0.15</td>
<td>0.4746</td>
</tr>
<tr>
<td>16</td>
<td>0.16</td>
<td>*</td>
</tr>
</tbody>
</table>

It can be seen from Table 3 that there are two feasible solutions of $\lambda_1^2$ denoted by $\lambda_1^2(+ \text{sign})$ and $\lambda_1^2(- \text{sign})$, gotten by using the “plus” and “minus” signs of (4.1) respectively. As a result, there are four distinct values of $\lambda_1$. They are:

$$
\lambda_{11} = +\sqrt{\lambda_1^2(+ \text{sign})}, \quad \lambda_{12} = -\sqrt{\lambda_1^2(+ \text{sign})}, \quad \lambda_{21} = +\sqrt{\lambda_1^2(- \text{sign})},$$

$$
\lambda_{22} = -\sqrt{\lambda_1^2(- \text{sign})},
$$

as suggested by Figure 1.

**COVARIANCE ANALYSIS OF SQUARES PDB(1) AND MA(1) PROCESSES**

First we establish that the square of the PDB(1) process identifies as an...
ARMA(1,1) process. Squaring model (2.1), we obtain

$$Y_t = X_t^2 = \theta_1 X_{t-1}^2 + \theta_2 X_{t-1} e_{t-1} e_t + \epsilon_t^2$$  \hspace{1cm} (5.1)

The first and second moments of (5.1) are:

$$E(Y_t) = E(X_t^2) = \frac{\sigma_1^2 \left(1 + 2\sigma_1^2 \theta_1^2\right)}{1 - \sigma_1^2 \theta_1^2} \ , \quad \sigma_1^2 \theta_1^2 < 1$$  \hspace{1cm} (5.2)

and for \(\sigma_1^2 \theta_1^2 < 1 \), \(3\sigma_1^4 \theta_1^4 < 1\)

$$R_y(k) = \begin{cases} 
2\sigma_1^4 \left(1 + 4\sigma_1^2 \theta_1^2 + 40\sigma_1^4 \theta_1^4 + 18\sigma_1^6 \theta_1^6 - 54\sigma_1^8 \theta_1^8\right), & k = 0 \\
(1 - \sigma_1^2 \theta_1^2)^2 \left(1 - 3\sigma_1^4 \theta_1^4\right), & k = 1 \ (5.3) \\
\sigma_1^2 \theta_1^2 R_y(k-1), & k \geq 2 
\end{cases}
$$

$$\rho_y(k) = \begin{cases} 
1, & k = 0 \\
\frac{3\sigma_1^2 \theta_1^2 \left(2 + 3\sigma_1^2 \theta_1^2 + 5 \sigma_1^4 \theta_1^4 + \sigma_1^6 \theta_1^6 - 8\sigma_1^8 \theta_1^8\right)}{1 + 4\sigma_1^2 \theta_1^2 + 40\sigma_1^4 \theta_1^4 + 18\sigma_1^6 \theta_1^6 - 54\sigma_1^8 \theta_1^8}, & k = 1 \ (5.4) \\
\sigma_1^2 \theta_1^2 \rho_y(k-1), & k \geq 2 
\end{cases}
$$

Equations (5.3) and (5.4) show that \(Y_t = X_t^2\) is a non-zero mean ARMA(1,1) process. Thus, \(Y_t = X_t^2\) can be represented as

$$Y_t = a + bY_{t-1} + cY_{t-1} + \epsilon_t$$  \hspace{1cm} (5.5)

where \(\epsilon_t\), \(t \in \mathbb{Z}\) may not be a purely random process with \(E(\epsilon_t) = 0\) and \(E(\epsilon_t^2) = \sigma_3^2 < \infty\).

The first and second moments of (5.5) are (Box et al., 1994):

$$E(Y_t) = \frac{a}{1 - b}$$  \hspace{1cm} (5.6)
Next we establish the relationship between the parameters of the square of a PDB(1) process and its equivalent ARMA(1,1) process by computing the values of $a$, $b$, $c$ and $\sigma$ for given $\theta_1$ and $\sigma_1$. We would use the method of moments as follows:

**Equating Means:**

$$\frac{\sigma_i^2 \left(1 + 2 \sigma_i^2 \theta_i^2 \right)}{1 - \sigma_i^2 \theta_i^2} = \frac{\sigma_i^2 \left(1 + 2 \lambda_i^2 \right)}{1 - \lambda_i^2} = \frac{a}{1 - b} \quad (5.9)$$

**Equating Variances:**

$$\frac{2 \sigma_i^4 \left(1 + 4 \sigma_i^2 \theta_i^2 + 40 \sigma_i^4 \theta_i^2 + 18 \sigma_i^6 \theta_i^6 - 54 \sigma_i^8 \theta_i^8 \right)}{\left(1 - \sigma_i^2 \theta_i^2 \right)^2 \left(1 - 3 \sigma_i^4 \theta_i^4 \right)} = \frac{\sigma_i^2 \left(1 + 2bc + c^2 \right)}{1 - b^2} \quad (5.10)$$

**Equating first order autocorrelations:**

$$\frac{3 \sigma_i^2 \theta_i^2 \left(2 + 3 \sigma_i^2 \theta_i^2 + 5 \sigma_i^4 \theta_i^4 + \sigma_i^6 \theta_i^6 - 8 \sigma_i^8 \theta_i^8 \right)}{1 + 4 \sigma_i^2 \theta_i^2 + 40 \sigma_i^4 \theta_i^4 + 18 \sigma_i^6 \theta_i^6 - 54 \sigma_i^8 \theta_i^8} = \frac{(b + c)(1 + bc)}{1 + 2bc + c^2} \quad (5.11)$$

**Equating second order autocorrelations:**

$$\sigma_i^2 \theta_i^2 \rho(1) = b \rho(1) \quad (5.12)$$

From (5.12) we obtain

$$b = \sigma_i^2 \theta_i^2 = \lambda_i^2 \quad (5.13)$$

Using (5.9) and substituting $b = \sigma_i^2 \theta_i^2 = \lambda_i^2$, we obtain

$$a = \sigma_i^2 \left(1 + 2 \sigma_i^2 \theta_i^2 \right) = \sigma_i^2 \left(1 + 2 \lambda_i^2 \right) \quad (5.14)$$

$$R_y(k) = \begin{cases} \frac{2\sigma^2_3 \left(1 + 2bc + c^2 \right)}{1 - b^2}, & |b| < 1, \quad k = 0 \\ \frac{\sigma^2_3 \left(b + c \right) \left(1 + bc \right)}{1 - b^2}, & |b| < 1, \quad k = 0 \\ bR_y(k - 1), & k \geq 2 \end{cases} \quad (5.7)$$

$$\rho_y(k) = \begin{cases} 1, & k = 0 \\ \frac{(b + c)(1 + bc)}{1 + 2bc + c^2}, & k = 1 \\ b\rho_y(k - 1), & k \geq 2 \end{cases} \quad (5.8)$$
If we let
\[
\begin{align*}
\begin{aligned}
d &= \frac{3\sigma_i^2 \theta_i^2 \left(2 + 3\sigma_i^2 \theta_i^2 + 5\sigma_i^4 \theta_i^4 + \sigma_i^6 \theta_i^6 - 8\sigma_i^8 \theta_i^8 \right)}{1 + 4\sigma_i^2 \theta_i^2 + 40\sigma_i^4 \theta_i^4 + 18\sigma_i^6 \theta_i^6 - 54\sigma_i^8 \theta_i^8} \\
&= \frac{3\lambda_i^2 \left(2 + 3\lambda_i^2 + 5\lambda_i^4 + \lambda_i^6 - 8\lambda_i^8 \right)}{1 + 4\lambda_i^2 + 40\lambda_i^4 + 18\lambda_i^6 - 54\lambda_i^8} = \rho \gamma (1)
\end{aligned}
\end{align*}
\]
we obtain from (5.11) that
\[
\begin{align*}
c &= \frac{(1-2d \sigma_i^2 \theta_i^2 + \sigma_i^4 \theta_i^4) \pm \sqrt{(1-2d \sigma_i^2 \theta_i^2 + \sigma_i^4 \theta_i^4)^2 - 4(d - \sigma_i^2 \theta_i^2)^2}}{2(d - \sigma_i^2 \theta_i^2)} \\
&\quad = \frac{(1-2d \lambda_i^2 + \lambda_i^4) \pm \sqrt{(1-2d \lambda_i^2 + \lambda_i^4)^2 - 4(d - \lambda_i^2)^2}}{2(d - \lambda_i^2)}
\end{align*}
\]
Using (5.10) we obtain
\[
\begin{align*}
\sigma_3^2 &= \frac{2\sigma_i^4 \left(1+4\sigma_i^2 \theta_i^2 + 40\sigma_i^4 \theta_i^4 + 18\sigma_i^6 \theta_i^6 - 54\sigma_i^8 \theta_i^8 \right)(1-b^2)}{(1-\sigma_i^2 \theta_i^2)^2 \left(1 - 3\sigma_i^4 \theta_i^4 \right)(1+2bc+c^2)} \\
&\quad = \frac{2\sigma_i^4 \left(1+4\lambda_i^2 + 40\lambda_i^4 + 18\lambda_i^6 - 54\lambda_i^8 \right)(1-b^2)}{(1-\lambda_i^2)^2 \left(1 - 3\lambda_i^4 \right)(1+2bc+c^2)}
\end{align*}
\]
Simulations concerning (5.16) reveals that for all $|\lambda_i| < 1$
\[
\frac{(1-2d \lambda_i^2 + \lambda_i^4) \pm \sqrt{(1-2d \lambda_i^2 + \lambda_i^4)^2 - 4(d - \lambda_i^2)^2}}{2(d - \lambda_i^2)} > 2
\]
and
\[
\frac{(1-2d \lambda_i^2 + \lambda_i^4) - \sqrt{(1-2d \lambda_i^2 + \lambda_i^4)^2 - 4(d - \lambda_i^2)^2}}{2(d - \lambda_i^2)} < 1
\]
Considering the invertibility condition, $|c| < 1$, it is clear that the feasible solution of $c$ is
\[ c = \frac{(1 - 2d \lambda_i^2 + \lambda_i^4)}{2(d - \lambda_i^2)} - \sqrt{\left(1 - 2d \lambda_i^2 + \lambda_i^4\right)^2 - 4\left(d - \lambda_i^2\right)^2} \] (5.20)

It is clear from (5.14) and (5.17) that \( a \) and \( \sigma^2 \) have different values for fixed \( \lambda = \sigma \theta \) due to the variations in \( \sigma \) and \( \theta \) while (5.13), (5.15) and (5.20) show that for fixed \( \lambda = \sigma \theta \), \( b, c \) and \( d \) are constants independent of the variations in \( \sigma \) and \( \theta \). The computations of \( b, c \) and \( d \) for fixed \( \lambda = \sigma \theta \) are given in Table 4. For want of space, Table 4 is an abridged table; the full table has \( \lambda = [-0.99, -0.98, ..., 0.98, 0.99] \) which is available in Ohakwe (2008).

Finally, we show that the square of an MA(1) process also identifies as an MA(1) process. Squaring model (2.2), we obtain the following results:

\[ W_t = X_t^2 = \beta_0^2 + \beta_1^2 u_{t-1}^2 + u_t^2 + 2 \beta_0 \beta_1 u_{t-1} + 2 \beta_0 u_t + 2 \beta_1 u_{t-1} u_t, \] (5.21)

\[ E(W_t) = \beta_0^2 + \sigma_2^2 \left(1 + \beta_1^2\right) \] (5.22)

\[ R_w(k) = \begin{cases} 
2\sigma_2^4 \left(1 + \beta_1^2\right)^2 + 4\beta_0^2 \sigma_2^2 \left(1 + \beta_1^2\right), & k = 0 \\
4\beta_0^2 \beta_1 \sigma_2^2 + 2 \beta_1^2 \sigma_2^4, & k = 1 \\
0, & k \geq 2 
\end{cases} \] (5.23)

\[ \rho_w(k) = \begin{cases} 
1, & k = 0 \\
\frac{4\beta_0^2 \beta_1 \sigma_2^2 + 2 \beta_1^2 \sigma_2^4}{2\sigma_2^4 \left(1 + \beta_1^2\right)^2 + 4\beta_0^2 \sigma_2^2 \left(1 + \beta_1^2\right)}, & k = 1 \\
0, & k \geq 2 
\end{cases} \] (5.24)

By looking at (2.8) and (5.24), it is clear that the covariance structure of models (2.2) and (5.21) are similar; thus the square of an MA(1) process identifies equally as an MA(1) process.
Table 4: Computations of the parameters of an ARMA(1,1) equivalent of the squares of a PDB(1) process.

<table>
<thead>
<tr>
<th>S/NO.</th>
<th>$\lambda_1$</th>
<th>$d = \rho_1(1)$</th>
<th>$b$</th>
<th>$c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.90</td>
<td>0.6938</td>
<td>0.8100</td>
<td>-0.2299</td>
</tr>
<tr>
<td>2</td>
<td>-0.80</td>
<td>0.6015</td>
<td>0.6400</td>
<td>-0.0604</td>
</tr>
<tr>
<td>3</td>
<td>-0.70</td>
<td>0.5498</td>
<td>0.4900</td>
<td>0.0859</td>
</tr>
<tr>
<td>4</td>
<td>-0.60</td>
<td>0.5203</td>
<td>0.3600</td>
<td>0.2228</td>
</tr>
<tr>
<td>5</td>
<td>-0.50</td>
<td>0.5000</td>
<td>0.2500</td>
<td>0.3441</td>
</tr>
<tr>
<td>6</td>
<td>-0.40</td>
<td>0.4630</td>
<td>0.1600</td>
<td>0.4009</td>
</tr>
<tr>
<td>7</td>
<td>-0.30</td>
<td>0.3684</td>
<td>0.0900</td>
<td>0.3272</td>
</tr>
<tr>
<td>8</td>
<td>-0.20</td>
<td>0.2085</td>
<td>0.0400</td>
<td>0.1764</td>
</tr>
<tr>
<td>9</td>
<td>-0.10</td>
<td>0.0583</td>
<td>0.0100</td>
<td>0.0485</td>
</tr>
<tr>
<td>10</td>
<td>0.00</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>11</td>
<td>0.10</td>
<td>0.0583</td>
<td>0.0100</td>
<td>0.0485</td>
</tr>
<tr>
<td>12</td>
<td>0.20</td>
<td>0.2085</td>
<td>0.0400</td>
<td>0.1764</td>
</tr>
<tr>
<td>13</td>
<td>0.30</td>
<td>0.3684</td>
<td>0.0900</td>
<td>0.3272</td>
</tr>
<tr>
<td>14</td>
<td>0.40</td>
<td>0.4630</td>
<td>0.1600</td>
<td>0.4009</td>
</tr>
<tr>
<td>15</td>
<td>0.50</td>
<td>0.5000</td>
<td>0.2500</td>
<td>0.3441</td>
</tr>
<tr>
<td>16</td>
<td>0.60</td>
<td>0.5203</td>
<td>0.3600</td>
<td>0.2228</td>
</tr>
<tr>
<td>17</td>
<td>0.70</td>
<td>0.5498</td>
<td>0.4900</td>
<td>0.0859</td>
</tr>
<tr>
<td>18</td>
<td>0.80</td>
<td>0.6015</td>
<td>0.6400</td>
<td>-0.0604</td>
</tr>
<tr>
<td>19</td>
<td>0.90</td>
<td>0.6938</td>
<td>0.8100</td>
<td>-0.2299</td>
</tr>
</tbody>
</table>

**CONCLUSION**

The fundamental truth about a PDB(1) process and an MA(1) process is that their covariance structure are similar which often times will lead to misclassification of one process as the other. However, within this similarity in covariance structure are some hidden differences between the two processes. Given the PDB(1) process, we obtained that $0 < \rho_1 < 0.16$ while for the MA(1) process, $|\rho_1| < 0.5$. In effect, any comparison of the two competing models must be done when $0 < \rho_1 < 0.16$.

Another glaring difference between the two is the fact that while the PDB(1) process with respect to the fourth moments identifies as an ARMA(1,1) process, the MA(1) process still identifies as an MA(1) process. In considering these moments, we have provided tables that show the relationship between the parameters of a given process and the corresponding parameters of the equivalent process.
REFERENCES


Iwueze, I. S. and D.C. Chikezie, (2006). Cumulant structure for the bilinear multiplicative seasonal $ARIMA(0, d, 0 \times (1, D, 1)$ time series model. Global Journal of Mathematical Sciences, 5(1), 25-34.


