A Note on the Variance of Generalized First Order Autoregressive Processes with Moving Average Errors

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Abstract

A new class of time series models known as Generalised Autoregressive of order one with first order moving average errors has been introduced in order to reveal some hidden features of certain time series data. A closed form of the variance of the process is derived. It is shown that in special cases this new result reduces to the standard ARMA results.

**Keywords:** Autoregression, Moving average, Errors, Autocorrelations, Variance, Autocovariance, Spectral density, Estimation, Time series, Fractional differencing, Long memory.
1 Introduction

It is known that the modelling of time series with changing frequency components is important in many applications, especially in financial data. Although ARMA type models could be used in practice, there is no systematic approach or suitable class of time series models available in literature to accommodate, analyze and forecast such time series. However, in recent years much attention has been focused on the study of fractional differencing and long memory time series to accommodate low frequency behaviour of a time series. In order to combine all such properties, slowly decaying autocorrelations and/or changing frequencies of a time series, a new class of generalized autoregressive model of order one (GAR(1)) has been introduced by Peiris (2003). This is a natural extension of the standard AR(1) model by adding an additional parameter \( \delta (>0) \) and is defined by,

\[
(1 - \alpha B)^\delta X_t = Z_t, \ |\alpha| < 1, \tag{1}
\]

where \( \{Z_t\} \sim WN(0, \sigma^2) \).

The autocovariance function of \( \text{GAR}(1) \) model in (1) is given by,

\[
\gamma_k = \frac{\sigma^2 \alpha^k \Gamma(k + \delta) F(\delta, k + \delta; k + 1; \alpha^2)}{\Gamma(\delta) \Gamma(k + 1)}, \quad k \geq 0, \tag{2}
\]

where \( F \) is the hypergeometric function defined as,

\[
F(a, b; c; z) = 1 + \frac{ab}{1!c} z + \frac{a(a + 1)b(b + 1)}{2!c(c + 1)} z^2 + \frac{a(a + 1)(a + 2)b(b + 1)(b + 2)}{3!c(c + 1)(c + 2)} z^3 + \ldots,
\]

or equivalently

\[
F(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{j=0}^{\infty} \frac{\Gamma(a + j)\Gamma(b + j)}{\Gamma(j + 1)\Gamma(c + j)} z^j. \tag{3}
\]

It has been shown by Peiris (2003) that the model in (1) could be used to model long memory and nearly long memory time series using suitably chosen \( \alpha \) and \( \delta \). Shitan and Peiris (2008) have studied the estimation problem that arise in class (1) using both the maximum likelihood (MLE) and
Whittle procedures. Further Peiris et al. (2004) have introduced the class of generalized moving average of order 1 (GMA(1)) which is given by,

$$X_t = (1 - \beta B)^\delta Z_t.$$  \hfill (4)

The corresponding autocovariance function of the GMA(1) model is

$$\gamma_k = \frac{\sigma^2 \beta^k \Gamma(k - \delta)F(-\delta, k - \delta; k + 1; \beta^2)}{\Gamma(-\delta)\Gamma(k + 1)}, \quad k \geq 0.$$  \hfill (5)

It has been shown that the additional parameter $\delta$ plays an important role in modelling and forecasting. We therefore refer the above classes as GAR(1;\delta) and GMA(1;\delta) for later reference. Peiris and Thavaneswaran (2007) and Peiris et al (2008) have shown that the classes given in (1) and (4) can be used to model many time series in practice, especially in finance.

A natural generalization of the standard ARMA (1,1) using the same approach is called Generalized Autoregressive Moving Average model (GARMA). This is denoted by GARMA (1, 1 ; $\delta_1$, $\delta_2$) and is defined by,

$$(1 - \alpha B)^{\delta_1}X_t = (1 - \beta B)^{\delta_2}Z_t,$$  \hfill (6)

where $-1 < \alpha, \beta < 1$, $\delta_1 > 0$ and $\delta_2 > 0$.

It is clear that (1) and (4) are special cases of (6). However, this paper considers a special case of (6) with $\delta_2 = 1$ (i.e. GARMA (1, 1 ; $\delta$, 1)) which is given as,

$$(1 - \alpha B)^\delta X_t = (1 - \beta B)Z_t.$$  \hfill (7)

It is easy to show that ( see, for example Peiris(2003) ) the process in (7) has a valid second order stationary solution such that,

$$X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j},$$

where $\psi(B) = (1 - \alpha B)^{-\delta}(1 - \beta B)$. Further, the corresponding spectral density function of the process $\{X_t\}$ is,

$$f(\omega) = \frac{\sigma^2(1 - 2\beta \cos \omega + \beta^2)}{2\pi(1 - 2\alpha \cos \omega + \alpha^2)^{\delta}}, \quad -\pi \leq \omega \leq \pi.$$  

The variance of this model is unknown and in this paper we provide an expression for it. We will also show that this expression will reduce to the variance of the standard ARMA models in special cases.
2 The Theoretical Variance of the GARMA(1, 1; δ, 1) Process

The variance $\gamma_0$ of the GARMA(1, 1; δ; 1) process is given in the following proposition.

**Proposition**

For the model defined as in equation (7), the variance $\gamma_0$ is given as,

$$
\gamma_0 = \frac{\sigma^2}{(1 + \alpha)^2 \delta_1} \left[ (1 + \beta)^2 F \left( \frac{1}{2}, \delta_1; 1; \frac{4\alpha}{(1 + \alpha)^2} \right) - 2\beta F \left( \frac{3}{2}, \delta_1; 2; \frac{4\alpha}{(1 + \alpha)^2} \right) \right],
$$

(8)

**Proof:**

We establish the above proposition by integrating the spectral density.

$$
\gamma_0 = \int_{-\pi}^{\pi} f(\omega) d\omega \\
= 2 \int_{0}^{\pi} f(\omega) d\omega \\
= \frac{\sigma^2}{\pi} \int_{0}^{\pi} \frac{(1 - 2\beta \cos \omega + \beta^2)}{(1 - 2\alpha \cos \omega + \alpha^2) \delta_1} d\omega \\
= \frac{\sigma^2}{(1 + \alpha)^2 \delta_1} \left[ (1 + \beta)^2 F \left( \frac{1}{2}, \delta_1; 1; \frac{4\alpha}{(1 + \alpha)^2} \right) - 2\beta F \left( \frac{3}{2}, \delta_1; 2; \frac{4\alpha}{(1 + \alpha)^2} \right) \right],
$$

which completes the proof.

3 Special Cases

In this section we verify the expression given in (8) by applying it to various special cases of GARMA(1, 1; δ; 1).

3.1 White Noise Model

If we set $\delta = 1, \alpha = 0, \beta = 0$ in the model given in equation (7), we get the white noise model, $X_t = Z_t$, whose variance is, $\gamma_0 = \sigma^2$. Substituting the
values \( \delta = 1, \alpha = 0, \beta = 0 \) in the expression in (8), we get

\[
\gamma_0 = \sigma^2 \left[ F \left( \frac{1}{2}, 1; 1; 0 \right) \right] = \sigma^2,
\]

since \( F \left( \frac{1}{2}, 1; 1; 0 \right) = 1 \) and this agrees with the variance for white noise model.

3.2 Moving Average MA(1) Model

When we set \( \delta = 1, \alpha = 0 \) in the model given in equation (7), we get the MA(1) model, \( X_t = (1 - \beta B)Z_t \), whose variance is, \( \gamma_0 = \sigma^2(1 + \beta^2) \). Substituting the values \( \delta = 1, \alpha = 0 \) in the expression in (8), we get

\[
\gamma_0 = \sigma^2 \left[ (1 + \beta)^2 F \left( \frac{1}{2}, 1; 1; 0 \right) - 2\beta F \left( \frac{3}{2}, 1; 2; 0 \right) \right]
\]

\[
= \sigma^2 \left[ (1 + \beta)^2 - 2\beta \right]
\]

\[
= \sigma^2(1 + \beta^2), \tag{9}
\]

which agrees with the variance for MA(1) model. Equation (9) is true since \( F \left( \frac{3}{2}, 1; 1; 0 \right) = 1 \).

3.3 Autoregressive AR(1) Model

On the other hand if we set \( \delta = 1, \beta = 0 \) in the model given in equation (7), we get the AR(1) model, \( (1 - \alpha B)X_t = Z_t \), whose variance is, \( \gamma_0 = \sigma^2/(1 - \alpha^2) \). Substituting the values \( \delta = 1, \beta = 0 \) in the expression in (8), we get

\[
\gamma_0 = \frac{\sigma^2}{(1 + \alpha)^2} \left[ F \left( \frac{1}{2}, 1; \frac{4\alpha}{(1 + \alpha)^2} \right) \right]. \tag{10}
\]

To evaluate

\[
F \left( \frac{1}{2}, 1; \frac{4\alpha}{(1 + \alpha)^2} \right),
\]

we make use of the following identity (see Gradshteyn and Ryzhik, 1994, page 1070, 9.134(2)),

\[
F \left( 2a, 2a + 1 - c; c; z \right) = \frac{1}{(1 + z)^{2a}} F \left( a, a + \frac{1}{2}; c; \frac{4z}{(1 + z)^2} \right), \tag{11}
\]

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where we substitute in (11) $a = \frac{1}{2}, c = 1, z = \alpha$ to get,

$$F(1, 1; 1; \alpha) = \frac{1}{(1 + \alpha)} F\left(\frac{1}{2}, 1; 1; \frac{4\alpha}{(1 + \alpha)^2}\right).$$

Therefore we have,

$$F\left(\frac{1}{2}, 1; 1; \frac{4\alpha}{(1 + \alpha)^2}\right) = (1 + \alpha)F(1, 1; 1; \alpha) = \frac{1 + \alpha}{1 - \alpha}. \quad (12)$$

Hence by substituting (12) into (10), we obtain

$$\gamma_0 = \frac{\sigma^2}{(1 + \alpha)^2} \left[1 + \frac{(\alpha - \beta)^2}{(1 - \alpha)^2}\right] = \frac{\sigma^2}{(1 - \alpha^2)},$$

which agrees with the variance for AR(1) model.

### 3.4 ARMA(1, 1) Model

For the ARMA(1, 1) model, we set $\delta = 1$ in the model given in equation (7), to get $(1 - \alpha B)X_t = (1 - \beta B)Z_t$. The variance of ARMA(1, 1) model is given as,

$$\gamma_0 = \frac{\sigma^2}{(1 + \alpha)^2} \left[1 + \frac{(\alpha - \beta)^2}{(1 - \alpha)^2}\right]. \quad (13)$$

Substituting the value $\delta = 1$ in the expression in (8), we get

$$\gamma_0 = \frac{\sigma^2}{(1 + \alpha)^2} \left[(1 + \beta)^2 F\left(\frac{1}{2}, 1; 1; \frac{4\alpha}{(1 + \alpha)^2}\right) - 2\beta F\left(\frac{3}{2}, 1; 2; \frac{4\alpha}{(1 + \alpha)^2}\right)\right]. \quad (14)$$

To evaluate

$$F\left(\frac{3}{2}, 1; 2; \frac{4\alpha}{(1 + \alpha)^2}\right),$$

we make use of the identity (see Gradshteyn and Ryzhik, 1994, page 1070, 9.134(3)),

$$F\left(a, a + \frac{1}{2} - b; b + \frac{1}{2}; z^2\right) = \frac{1}{(1 + z)^{2a}} F\left(a, b; 2b; \frac{4z}{(1 + z)^2}\right), \quad (15)$$
and put $a = \frac{3}{2}, b = 1, z = \alpha$ to get,

$$F \left( \frac{3}{2}, 1; \frac{3}{2}; \alpha^2 \right) = \frac{1}{(1 + \alpha)^3} F \left( \frac{3}{2}, 1; \frac{3}{2}; \frac{4\alpha}{(1 + \alpha)^2} \right).$$

Hence,

$$F \left( \frac{3}{2}, 1; 2; \frac{4\alpha}{(1 + \alpha)^2} \right) = (1 + \alpha)^3 F \left( \frac{3}{2}, 1; \frac{3}{2}; \alpha^2 \right). \quad (16)$$

Using (see Abramowitz and Stegun (1964), 15.2.19, page 558),

$$(b - a)(1 - z) F(a, b; c; z) - (c - a) F(a - 1, b; c; z) + (c - b) F(a, b - 1; c; z) = 0. \quad (17)$$

By setting $a = \frac{3}{2}, b = 1, c = \frac{3}{2}$ and $z = \alpha^2$ in equation (17), we obtain

$$\left( -\frac{1}{2} \right) \left( 1 - \alpha^2 \right) F \left( \frac{3}{2}, 1; \frac{3}{2}; \alpha^2 \right) - 0 + \frac{1}{2} F \left( \frac{3}{2}, 0; \frac{3}{2}; \alpha^2 \right) = 0$$

$$\left( -\frac{1}{2} \right) \left( 1 - \alpha^2 \right) F \left( \frac{3}{2}, 1; \frac{3}{2}; \alpha^2 \right) + \frac{1}{2} (1) = 0$$

$$F \left( \frac{3}{2}, 1; \frac{3}{2}; \alpha^2 \right) = \frac{1}{(1 - \alpha^2)} \quad (18)$$

By substituting (18) into equation (16) we obtain

$$F \left( \frac{3}{2}, 1; 2; \frac{4\alpha}{(1 + \alpha)^2} \right) = (1 + \alpha)^3 \frac{1}{1 - \alpha^2} \quad (19)$$

Now (14) gives,

$$\gamma_0 = \frac{\sigma^2}{(1 + \alpha)^2} \left[ (1 + \beta)^2 \frac{(1 + \alpha)}{1 - \alpha} - 2\beta \frac{(1 + \alpha)^3}{1 - \alpha^2} \right]$$

$$= \sigma^2 \left[ \frac{(1 + \beta)^2}{(1 + \alpha)(1 - \alpha)} - \frac{2\beta(1 + \alpha)}{1 - \alpha^2} \right]$$

$$= \frac{\sigma^2}{1 - \alpha^2} \left[ (1 + \beta)^2 - 2\beta(1 + \alpha) \right]$$

$$= \frac{\sigma^2}{1 - \alpha^2} \left[ 1 + 2\beta + \beta^2 - 2\beta - 2\alpha\beta \right]$$

$$= \frac{\sigma^2}{1 - \alpha^2} \left[ 1 - 2\alpha\beta + \beta^2 \right]$$
\[
\frac{\sigma^2}{1 - \alpha^2} \left[ 1 - \alpha^2 + \alpha^2 - 2\alpha\beta + \beta^2 \right] \\
= \frac{\sigma^2}{1 - \alpha^2} \left[ (1 - \alpha^2) + (\alpha - \beta)^2 \right] \\
= \sigma^2 \left[ 1 + \frac{(\alpha - \beta)^2}{(1 - \alpha^2)} \right],
\]

which agrees with the variance of ARMA(1, 1) as given in equation (13).

4 Conclusion

The objective of this paper was to establish the variance of the GARMA(1, 1; \delta, 1) process and the result is contained in the proposition. We have also sucessfully verified that our proposition reduces to many special cases considered in section 3. This result contributes to the theory of the model considered in this paper.

Acknowledgement

I would like to thank the referee and the editor for their useful comments and valuable suggestions to improve the quality of the paper.

References


