MAXIMUM LIKELIHOOD ESTIMATION AND OPTIMAL DESIGN IN STEP PARTIALLY ACCELERATED LIFE TESTS FOR THE PARETO DISTRIBUTION OF THE SECOND KIND WITH TYPE-I CENSORING

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ABSTRACT

This paper deals with simple time-step stress Partially Accelerated Life Tests (PALT) where a pre-specified censoring time is involved. The time to failure is assumed to have a two-parameter Pareto lifetime distribution of the second kind. Maximum likelihood estimates (MLE) of the PALT model parameters are obtained. In addition, confidence intervals estimation for the parameters is presented. Optimum plans for simple step-stress PALT are also considered. Such plans minimize the generalized asymptotic variance (GAV) of the MLE of the model parameters. For illustration, numerical examples are given.

1. INTRODUCTION

There are many problems of life testing which require a long time to acquire the test data at the specified use condition. In such problems, accelerated life tests (ALT) or partially accelerated life tests (PALT) are often used to shorten the lives of test items. In an ALT, test items are run only at accelerated conditions. But in a PALT, the test items can be run at both normal use and accelerated conditions. According to Nelson (1990), the stress can be applied in various ways, commonly used method is step-stress. The step-stress scheme allows the stress setting of a unit to be changed at pre-specified times or upon the occurrence of a fixed number of failures. In this paper, we consider a simple time step-stress (i.e., only two-step) PALT in which a test item is first run at use condition and, if it does not fail for a specified time τ, then it is run at accelerated condition until failure. The intent of such an experiment is to collect more failure data in a limited time without necessarily using a high stress to all test units. Bhattacharyya and Soejoeti (1989) indicated that step-stress PALT are practical for many problems of life testing where the test process requires a long time if the test is simply carried out under the use condition.

There is an amount of literature on such step-stress partially accelerated life tests. Goel (1971) considered the estimation problem of the acceleration factor β; which is the ratio of the hazard rate at accelerated condition to that at use condition; using the maximum

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likelihood method for items having exponential distribution and uniform distribution in the case of complete sampling. He also obtained the optimal PALT plan. DeGroot and Goel (1979) considered a PALT model in which a test item is first run at use condition and if it does not fail for a specified time τ, then it is run at accelerated condition until failure. That is, $Y = T$, if $T \leq \tau$; and $Y = \tau + \beta^{1} (T-\tau)$, if $T > \tau$; where T is the lifetime of an item at use condition and Y its total lifetime. Bhattcharyya and Soejoeti (1989) proposed a failure rate model in which, $h^{'}(y) = h(y)$ *if* $y \leq \tau$ and $h^{'}(y) = \beta h(y)$ if $y > \tau$ where $h(.)$ and $h^{'}(.)$ are failure-rate functions of T and Y, respectively. When T follows a Weibull distribution, Bhattcharyya and Soejoeti (1989) obtained the estimates of the model parameters by the maximum likelihood method in the case of complete sampling.

Also, PALT were studied with type-I censored data by few authors. Bai and Chung (1992) used the maximum likelihood method to estimate the scale parameter and the acceleration factor for exponentially distributed lifetime in the case of type-I censoring. They also considered the problem of optimally designing for the step PALT that terminate at a predetermined time. Bai, Chung and Chun (1993) considered the same work of Bai and Chung (1992) for items having a lognormal distribution. Abdel-Ghaly, Attia and Abdel-Ghani (1996) considered the maximum likelihood method for estimating the acceleration factor and the parameters of Weibull distribution in the case of type-I censoring.

In addition to this introductory section, the paper includes six sections. In Section 2, we introduce Pareto distribution of the second kind as a lifetime model. In Section 3, the maximum likelihood estimators of the acceleration factor and the parameters of Pareto distribution are obtained. Their Confidence limits are presented in Section 4. Section 5 discusses the optimal design of step-stress PALT. For illustration, simulation studies are given in Section 6. Finally, Section 7 presents the concluding remarks.

2. THE MODEL

Notation

n	total number of test items in a PALT
η	Censoring time of a PALT
T	lifetime of an item at use condition
Y	total lifetime of an item in a step PALT
$f(t)$	probability density function
$R(t)$	reliability function at time t at use condition
$H(t)$	Hazard (failure) rate at time t at use condition
β	Acceleration factor (β>1)
τ	stress change-time in a step PALT ($\tau < \eta$)
P_u	Probability that an item fails at use condition

P_a probability that an item fails at accelerated condition

$\wedge$ implies a maximum likelihood estimate

θ Pareto scale parameter

α Pareto shape parameter

Y_i Observed value of the total lifetime Y_i of item i , $I = 1, \ldots, n$

δ_{1i}, δ_{2i} Indicator functions: $\delta_{1i} \equiv I(Y_i \leq \tau)$, $\delta_{2i} \equiv I(\tau < Y_i \leq \eta)$

n_u, n_a Numbers of items failed at use and accelerated conditions, respectively

$y_{(1)} \leq \ldots\ldots \leq y_{(nu)} \leq \tau \leq y_{(nu+1)} \leq \ldots\ldots \leq y_{(nu+na)} \leq \eta$ ordered failure times.

The Pareto Distribution:

In this paper, we consider T a random variable denoting the lifetime of an item at use condition with Pareto density of the second kind,

$$f(t;\theta,\alpha) = \frac{\alpha\theta^{\alpha}}{(\theta+t)^{\alpha+1}} \quad ; t > 0,\ \theta > 0,\ \alpha > 0. \tag{1}$$

The reliability function of the Pareto distribution takes the form:

$$R(t) = \frac{\theta^{\alpha}}{(\theta+t)^{\alpha}} \quad ; t > 0,\ \theta > 0,\ \alpha > 0, \tag{2}$$

and the corresponding hazard rate is given by:

$$h(t) = \frac{\alpha}{\theta+t} \quad ; t > 0,\ \theta,\ \alpha > 0. \tag{3}$$

which is a decreasing function as $t > 0$, representing the early failure region or the initial failure region in the bathtub shape. This type of failures may be due to initial weakness or defects, weak parts, bad assembly or poor fits, etc. [see Martz, (1992)]. Therefore, Pareto distribution may be used as a reliability growth model. It is important to point out that the Pareto distribution has a wide use in economic studies. Many socio-economic phenomena such as city population sizes, occurrence of natural resources, stock price fluctuations, size of firms, and personal incomes are distributed according to certain statistical distribution with very long right tails. The Pareto distribution has played a major role in the investigation of previous phenomena [Johnson and Kotz, (1970)]. Smith (1989) applied Pareto distribution to the study of ozone levels in the upper atmosphere. Grimshaw (1993) used Pareto distribution to model tensile-strength data from a random sample of nylon carpet fibers. Howlader and Hossain, (2002) indicated that since the Pareto distribution has a decreasing hazard or failure rate, it has often been used to model incomes and survival times. Censored data also seems to occur often in these two applications.

3. MAXIMUM LIKELIHOOD ESTIMATION

The method of maximum likelihood (ML) has long been widely accepted as one of the most reliable methods of estimating distribution parameters. It has been commonly used in the analysis of accelerated life tests. Although the exact sampling distribution of maximum likelihood estimators (MLE) is sometimes unknown, MLE have the desirable properties of being consistent, asymptotically normal and asymptotically efficient for large samples under appropriate regularity conditions. As indicated by Grimshaw (1993), the ML method is commonly used for most theoretical models and kinds of censored data. Also, Bugaighis (1988) pointed out that the maximum likelihood procedure generally yields efficient estimators. However, these estimators do not always exist in a closed form, so, numerical techniques are used to compute them.

The lifetime of the test item is assumed to follow the Pareto distribution with scale parameter θ and shape parameter α. Therefore, the probability density function of total lifetime Y of an item in a step PALT is given by:

$$f(y)=\begin{cases} 0 & y \le 0 \\ \dfrac{\alpha\theta^{\alpha}}{(\theta+y)^{\alpha+1}} & 0<y\le\tau \\ \dfrac{\beta\alpha\theta^{\alpha}}{\{\theta+\tau+\beta(y-\tau)\}^{\alpha+1}} & y>\tau \end{cases} \tag{4}$$

where $\theta > 0$ and $\alpha > 0$.

The observed values of the total lifetime Y are given by:

$$y_{(1)} \le \ldots\ldots \le y_{(nu)} \le \tau \le y_{(nu+1)} \le \ldots\ldots \le y_{(nu+na)} \le \eta$$

Let δ_{1i}, δ_{2i}, be indicator functions such that $\delta_{1i} \equiv I(Y_i \le \tau)$, and $\delta_{2i} \equiv I(\tau < Y_i \le \eta)$; where $i = 1,\ldots,n$. Since the total lifetimes $y_1, \ldots, y_n$ of n items are independent and identically distributed random variables, then the total likelihood function for them is given by:

$$L(\beta,\theta,\alpha,Y_i,\delta_{1i},\delta_{2i})=\prod_{i=1}^{n}[\frac{\alpha\theta^{\alpha}}{(\theta+y_i)^{\alpha+1}}]^{\delta_{1i}}.[\frac{\beta\alpha\theta^{\alpha}}{\{\theta+\tau+\beta(y_i-\tau)\}^{\alpha+1}}]^{\delta_{2i}} .[\frac{\theta^{\alpha}}{\{\theta+\tau+\beta(\eta-\tau)\}^{\alpha+1}}]^{\bar{\delta}_{1i}\bar{\delta}_{2i}} \tag{5}$$

$\bar{\delta}_{1i} = 1-\delta_{1i}$ and $\bar{\delta}_{2i} = 1-\delta_{2i}$

It is usually easier to maximize the natural logarithm of the likelihood function rather than the likelihood function itself. The natural logarithm of the likelihood function is given by:

$$\ln L = (n_u + n_a)\ln\alpha + n\alpha\ln\theta + n_a \ln\beta - (n - n_u - n_a)\alpha\ln\{\theta+\tau+\beta(\eta-\tau)\}$$
$$-(\alpha+1)[\sum_{i=1}^{n}\delta_{1i}\ln(\theta+y_i) + \sum_{i=1}^{n}\delta_{2i}\ln\{\theta+\tau+\beta(y_i-\tau)\}] \qquad (6)$$

The first derivatives of the natural logarithm of the total likelihood function in (6) with respect to β, θ and α are given by:

$$\frac{\partial \ln L}{\partial\beta} = \frac{n_a}{\beta} - \frac{(n-n_u-n_a)\alpha(\eta-\tau)}{\{\theta+\tau+\beta(\eta-\tau)\}} - (\alpha+1)\sum_{i=1}^{n}\delta_{2i}\frac{(y_i-\tau)}{\{\theta+\tau+\beta(y_i-\tau)\}} \qquad (7)$$

$$\frac{\partial \ln L}{\partial\theta} = \frac{n\alpha}{\theta} - \frac{(n-n_u-n_a)\alpha}{\{\theta+\tau+\beta(\eta-\tau)\}}$$
$$-(\alpha+1)[\sum_{i=1}^{n}\frac{\delta_{1i}}{(\theta+y_i)} + \sum_{i=1}^{n}\frac{\delta_{2i}}{\{\theta+\tau+\beta(y_i-\tau)\}}] \qquad (8)$$

$$\frac{\partial \ln L}{\partial\alpha} = \frac{n_u+n_a}{\alpha} + n\ln\theta - (n-n_u-n_a)\ln\{\theta+\tau+\beta(\eta-\tau)\}$$
$$-\sum_{i=1}^{n}\delta_{1i}\ln(\theta+y_i) - \sum_{i=1}^{n}\delta_{2i}\ln\{\theta+\tau+\beta(y_i-\tau)\} \qquad (9)$$

From equation (9) the maximum likelihood estimate of α is given by the following estimation equation:

$$\hat{\alpha} = \frac{(n_u+n_a)}{(n-n_u-n_a)\ln\{\hat{\theta}+\tau+\hat{\beta}(\eta-\tau)\} - n\ln\hat{\theta} + Q} \qquad (10)$$

where

$$Q = \sum_{i=1}^{n}\delta_{1i}\ln(\hat{\theta}+y_i) + \sum_{i=1}^{n}\delta_{2i}\ln\{\hat{\theta}+\tau+\hat{\beta}(y_i-\tau)\}$$

Therefore, by substituting for α into the two equations (7) and (8) and equating each of the resulting equations to zero, they are reduced to the following two non-linear estimating equations:

$$\frac{n_a}{\hat{\beta}} - \frac{(n_u + n_a)(n - n_u - n_a)(\eta - \tau)}{[(n - n_u - n_a)\ln\{\hat{\theta} + \tau + \hat{\beta}(\eta - \tau)\} - n\ln\hat{\theta} + Q]\{\hat{\theta} + \tau + \hat{\beta}(\eta - \tau)\}}$$
$$-\left(\frac{(n_u + n_a)}{(n - n_u - n_a)\ln\{\hat{\theta} + \tau + \hat{\beta}(\eta - \tau)\} - n\ln\hat{\theta} + Q} + 1\right)$$
$$\cdot\sum_{i=1}^{n} \delta_{2i} \frac{(y_i - \tau)}{\{\hat{\theta} + \tau + \hat{\beta}(y_i - \tau)\}} = 0 \qquad (11)$$

and,

$$\frac{n(n_u + n_a)}{\hat{\theta}(n - n_u - n_a)\ln\{\hat{\theta} + \tau + \hat{\beta}(\eta - \tau)\} - n\ln\hat{\theta} + Q}$$
$$-\frac{(n_u + n_a)(n - n_u - n_a)}{[(n - n_u - n_a)\ln\{\hat{\theta} + \tau + \hat{\beta}(\eta - \tau)\} - n\ln\hat{\theta} + Q]\{\hat{\theta} + \tau + \hat{\beta}(\eta - \tau)\}}$$
$$-\left(\frac{(n_u + n_a)}{(n - n_u - n_a)\ln\{\hat{\theta} + \tau + \hat{\beta}(\eta - \tau)\} - n\ln\hat{\theta} + Q} + 1\right)\left[\sum_{i=1}^{n} \frac{\delta_{1i}}{(\hat{\theta} + y_i)}\right.$$
$$\left. + \sum_{i=1}^{n} \frac{\delta_{2i}}{\{\hat{\theta} + \tau + \hat{\beta}(y_i - \tau)\}}\right] = 0 \qquad (12)$$

Since closed-form solutions to equations (11) and (12) are highly unlikely, iterative estimation procedures must be used to solve these equations. The Newton-Raphson procedure is applied for the simultaneous numerically solution of these nonlinear equations. Thus, once the values of β and θ are determined, an estimate of α is easily obtained from (10). In relation to the asymptotic variance-covariance matrix of the ML estimators of the parameters, it can be approximated by numerically inverting the Fisher-information matrix composed of the negative second derivatives of the natural logarithm of the likelihood function evaluated at the ML estimates. The asymptotic Fisher-information matrix can be written as follows:

$$F = \begin{bmatrix} -\frac{\partial^2 \ln L}{\partial\beta^2} & -\frac{\partial^2 \ln L}{\partial\beta\partial\theta} & -\frac{\partial^2 \ln L}{\partial\beta\partial\alpha} \\ -\frac{\partial^2 \ln L}{\partial\theta\partial\beta} & -\frac{\partial^2 \ln L}{\partial\theta^2} & -\frac{\partial^2 \ln L}{\partial\theta\partial\alpha} \\ -\frac{\partial^2 \ln L}{\partial\alpha\partial\beta} & -\frac{\partial^2 \ln L}{\partial\alpha\partial\theta} & -\frac{\partial^2 \ln L}{\partial\alpha^2} \end{bmatrix} \downarrow (\hat{\beta}, \hat{\theta}, \hat{\alpha}) \qquad (13)$$

The elements of the above matrix F can be expressed by the following equations:

$$\frac{\partial^2 \ln L}{\partial\beta^2} = -\frac{n_a}{\beta^2} + \frac{(n - n_u - n_a)\alpha(\eta - \tau)^2}{\{\theta + \tau + \beta(\eta - \tau)\}^2} + (\alpha + 1)\sum_{i=1}^{n} \delta_{2i} \frac{(y_i - \tau)^2}{\{\theta + \tau + \beta(y_i - \tau)\}^2} \qquad (14)$$

$$\frac{\partial^2 \ln L}{\partial \theta^2} = -\frac{n\alpha}{\theta^2} + \frac{(n-n_u-n_a)\alpha}{\{\theta+\tau+\beta(\eta-\tau)\}^2}$$

$$+(\alpha+1)[\sum_{i=1}^{n} \frac{\delta_{1i}}{(\theta+y_i)^2} + \sum_{i=1}^{n} \frac{\delta_{2i}}{\{\theta+\tau+\beta(y_i-\tau)\}^2}] \tag{15}$$

$$\frac{\partial^2 \ln L}{\partial \alpha^2} = -\frac{n_u+n_a}{\alpha^2} \tag{16}$$

$$\frac{\partial^2 \ln L}{\partial \beta \partial \theta} = \frac{(n-n_u-n_a)\alpha(\eta-\tau)}{\{\theta+\tau+\beta(\eta-\tau)\}^2} + (\alpha+1)\sum_{i=1}^{n}\delta_{2i}\frac{(y_i-\tau)}{\{\theta+\tau+\beta(y_i-\tau)\}^2} \tag{17}$$

$$\frac{\partial^2 \ln L}{\partial \beta \partial \alpha} = -\frac{(n-n_u-n_a)(\eta-\tau)}{\{\theta+\tau+\beta(\eta-\tau)\}} - \sum_{i=1}^{n}\delta_{2i}\frac{(y_i-\tau)}{\{\theta+\tau+\beta(y_i-\tau)\}} \tag{18}$$

$$\frac{\partial^2 \ln L}{\partial \theta \partial \alpha} = \frac{n}{\theta} - \frac{(n-n_u-n_a)}{\{\theta+\tau+\beta(\eta-\tau)\}} - [\sum_{i=1}^{n} \frac{\delta_{1i}}{(\theta+y_i)} + \sum_{i=1}^{n} \frac{\delta_{2i}}{\{\theta+\tau+\beta(y_i-\tau)\}}] \tag{19}$$

Therefore, the maximum likelihood estimators of β, θ and α have an asymptotic variance-covariance matrix defined by inverting the above Fisher information matrix.

4. CONFIDENCE INTERVAL ESTIMATION

The maximum likelihood method provides a single point estimate for a population parameter. A confidence interval indicates the uncertainty in an estimate calculated from sample data, it encloses the population parameter with a specified high probability. The most common method to set confidence bounds for the parameters is to use the large-sample normal distribution of the ML estimates [Vander Wiel and Meeker, (1990)].

To define a confidence interval for a population parameter λ; suppose $L_\lambda = L_\lambda(y_1, \ldots, y_n)$ and $U_\lambda = U_\lambda(y_1, \ldots, y_n)$ are functions of the sample data $y_1, \ldots, y_n$ such that

$$P_\lambda(L_\lambda \le \lambda \le U_\lambda) = \gamma \tag{20}$$

Where the interval $[L_\lambda, U_\lambda]$ is called a two-sided γ100 % confidence interval for λ. L_λ, U_λ are the lower and upper confidence limits for λ, respectively. The random limits L_λ, U_λ enclose λ with probability γ.

For large sample size the maximum likelihood estimators, under appropriate regularity conditions, are consistent and asymptotically normally distributed. Therefore, the two-sided approximate $\gamma 100$ % confidence limits for a population parameter λ can be obtained such that

$$P[-z \leq \frac{\hat{\lambda}-\lambda}{\sigma(\hat{\lambda})} \leq z] \cong \gamma \tag{21}$$

where z is the $[100(1-\gamma/2]^{th}$ standard normal percentile. Therefore, the two-sided approximate $\gamma 100$ % confidence limits for β, θ and α are given , respectively, as follows:

$$\left.\begin{array}{ll} L_{\beta} = \hat{\beta} - z\sigma(\hat{\beta}) & U_{\beta} = \hat{\beta} + z\sigma(\hat{\beta}) \\ L_{\theta} = \hat{\theta} - z\sigma(\hat{\theta}) & U_{\theta} = \hat{\theta} + z\sigma(\hat{\theta}) \\ L_{\alpha} = \hat{\alpha} - z\sigma(\hat{\alpha}) & U_{\alpha} = \hat{\alpha} + z\sigma(\hat{\alpha}) \end{array}\right] . \tag{22}$$

5. OPTIMAL DESIGN

Optimal Stress Change-Time τ^*:

In this section, the problem of optimally designing a simple step-stress PALT, which terminates at a predetermined time, is considered. For the optimal design stage of the test, a new experiment with test units different from those tested in the stage of parameter estimation is conducted. It is worth noting that the stress change point τ is pre-specified for the stage of parameter estimation, but for the optimal design stage of the test τ is considered as a switching parameter and has to be optimally determined according to a certain optimally criterion.

The optimum test plan for products having a two-parameter Pareto lifetime distribution of the second kind is derived in which the choice of τ^* will be investigated such that the GAV of the MLE of the model parameters at normal use stress is minimized.

Generalized Asymptotic Variance of the MLE of the Model Parameters: an optimality criterion

The GAV of the MLE of the model Parameters is the reciprocal of the determinant of *F*. That is,

$$\text{GAV}(\hat{\beta},\hat{\theta},\hat{\alpha}) = |F|^{-1}$$

The Newton-Raphson method is applied to obtain the optimal stress-change time τ^* which minimize the GAV as defined above. Therefore, the corresponding optimal number of items failed at use and accelerated conditions are, respectively, given by:

$$nP_u = n[1 - \frac{(\hat{\theta})^{\hat{\alpha}}}{(\hat{\theta} + \tau^*)^{\hat{\alpha}}}]$$

and

$$nP_a = n[\frac{(\hat{\theta})^{\hat{\alpha}}}{(\hat{\theta} + \tau^*)^{\hat{\alpha}}}.(1 - \frac{(\hat{\theta})^{\hat{\alpha}}}{\{\hat{\theta} + \hat{\beta}(\eta - \tau^*)\}^{\hat{\alpha}}})]$$

6. SIMULATION STUDIES

In this section numerical results are given in which the parameters are estimated from simulated partially accelerated life test data drawn from Pareto distribution of the second kind. We use the Newton-Raphson iterative algorithm and programs written in Pascal language for finding the MLE of β and θ assuming initial values for these parameters. So, the derived nonlinear logarithmic likelihood estimating equations (11) and (12) are solved iteratively. Once the values of β and θ are determined, an estimate of the shape parameter α is easily obtained from (10). To explore the effect of sample size on the MLE, different sized samples were simulated ranging in size from n =100 to n =500 with results presented in both Table (1) and Table (2) with parameters set at β=2, θ=1, and α=2, given τ= 1.5 and η = 3. For each sample size of n =100(100)500, the experiment has 500 replications. Another set of data are generated with parameters set at β=4, θ=2, and α=4, given τ= 1.5 and η = 3. Also, 500 repeated samples are obtained randomly for each sample size of n = 100(100)500, with results shown in both Table (3) and Table (4).

Table (1) and Table (3) demonstrate the average numbers of items, n_u and n_a, failed at use and accelerated conditions, respectively. Also, both Table (1) and Table (3) summarize the results of solving the ML equations of β, θ and α for the different two sets of data, respectively. The numerical results indicate that the estimators approximate the true values of the parameters when the sample size n is increasing. Moreover, Table (1) and Table (3) show the asymptotic variances of the MLE of β, θ and α for different sized samples. As shown from the numerical results, the asymptotic variances of the estimators decrease as the sample size n is getting to be large. To construct the approximate confidence intervals for the three parameters β, θ and α, the equations in (22) are used for each parameter and five different sized samples of n = 100(100)500 are used from the two populations stated before with results shown in both Table (1) and Table (3). These Tables present 2-sided approximate confidence bounds, based on 95% confidence degree, for the parameters. As shown from the numerical results, the intervals of the parameters appear to be narrow as the sample size n increases.

For the two sets of data stated before, optimum test plans are also developed, numerically. Both Table (2) and Table (4) give the optimal stress change-time for the considered different sized samples. Also, the corresponding optimal numbers of items failed at use and accelerated conditions, nP_u and nP_a , respectively, are presented in these tables. The numerical results shown in both Table (2) and Table (4) demonstrate that the optimal stress change-time, minimizing the GAV of the MLE of the model parameters, usually approach the censoring time at which the life test is terminated. That is, all units tend to fail at use condition and no items appear to fail at accelerated conditions. Also, Table (2) and Table (4) show the optimal GAV of the MLE of the model parameters which is numerically obtained with τ^* in place of τ for different sized samples. As indicated from the results, the optimal GAV of the MLE of the model parameters decrease as the sample size n increases.

TABLE (1)

The ML estimates, estimated asymptotic variances of the ML estimators and confidence bounds of the parameters (β, θ, α) set at (2, 1, 2), respectively, given τ=1.5 and η=3 for different sized samples under type-I censoring

n	n_u	n_a	Parameter	Estimate	Variance	Lower bound	Upper bound
100	83	13	β	2.7943	2.0020	0.0211	5.5675
			θ	2.3353	0.9896	0.3855	4.2851
			α	2.9388	2.1095	0.0921	5.7855
200	167	26	β	2.0793	0.6020	0.5584	3.6001
			θ	1.8138	0.5391	0.3747	3.2529
			α	2.5722	1.1278	0.4907	4.6537
300	252	38	β	2.0187	0.3738	0.8203	3.2171
			θ	1.1414	0.2585	0.1448	2.1380
			α	2.2188	0.5915	0.7113	3.7262
400	336	51	β	2.0553	0.2837	1.0112	3.0994
			θ	1.0806	0.1515	0.3175	1.8436
			α	2.1200	0.3373	0.9817	3.2583
500	419	64	β	2.0444	0.2258	1.1129	2.9759
			θ	1.0891	0.1311	0.3791	1.7990
			α	2.1275	0.2929	1.0667	3.1884

TABLE (2)

The results of optimal design of the life test, given τ=1.5 and η=3, for different sized samples under type-I censoring

n	τ^*	np_u	nP_a	Optimal GAV
100	2.997453	96	0	0.027264
200	2.999915	193	0	0.001808
300	2.999921	290	0	0.000262
400	2.999924	387	0	0.000070
500	2.999923	483	0	0.000039

TABLE (3)
The ML estimates, estimated asymptotic variances of the ML estimators and confidence bounds of the parameters (β, θ, α) set at (4, 2, 4), respectively, given τ=1.5 and η=3 for different sized samples under type-I censoring

n	n_u	n_a	Parameter	Estimate	Variance	Lower bound	Upper bound
100	88	10	β	4.0762	3.5937	0.3606	7.7918
			θ	3.8929	3.0086	0.4932	7.2926
			α	5.7030	4.4160	1.5842	9.8218
200	177	21	β	4.0266	2.6314	0.8472	7.2061
			θ	3.6415	2.2247	0.7181	6.5649
			α	4.8288	3.6056	1.1071	8.5505
300	267	31	β	4.0245	1.7236	1.4513	6.5978
			θ	2.7519	1.8822	0.0630	5.4408
			α	4.3312	2.8983	0.9944	7.6680
400	357	41	β	4.0099	1.2704	1.8007	6.2191
			θ	2.4375	1.2608	0.2367	4.6383
			α	4.2219	1.7015	1.6652	6.7786
500	445	52	β	4.1439	1.0575	2.1284	6.1594
			θ	2.1108	0.9867	0.1639	4.0577
			α	4.1582	0.6246	2.6092	5.7072

TABLE (4)
The results of optimal design of the life test, given τ=1.5 and η=3, for different sized samples under type-I censoring

n	τ^*	nP_u	nP_a	Optimal GAV
100	2.8930	96	2	1.3832
200	2.9069	197	1	0.3795
300	2.9910	297	1	0.1174
400	2.9707	396	2	0.0394
500	2.9916	496	1	0.0210

7. CONCLUDING REMARKS

This paper considered the problems of estimation and optimally designing simple step-stress PALT for the Pareto distribution under type-I censoring. The maximum likelihood estimates of the model parameters were obtained. Also, optimal simple step-stress PALT plans were developed numerically under the assumptions of Pareto lifetimes of test units and type-I censoring. The optimality criterion adopted was the minimization of the GAV of the MLE of the model parameters. As shown from the numerical results, at different sized samples, τ^* always approach the censoring time η indicating that all items tend to fail at normal use condition, i.e., testing only at normal condition.

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