Comparison of Bayes’ Estimators Under Different Loss Functions of a 1-out-of-2:G Repairable System

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Abstract: In this paper we derive Bayes estimators of the mean life time and the Quasi Bayes’ reliability function of a 1-out-of-2:G repairable system. Three loss functions and two prior distributions have been considered. The statistical performances of the Bayes estimates relative to Quadratic loss function, Linex loss function and El-Sayyad loss function are compared through root mean squared error (RMSE) based on simulation study.

Keywords: Prior and posterior distribution, quasi density, loss functions, root mean square error.

1. Introduction:

Bayesian analysis combines the prior knowledge of the random variations in the parameters of the lifetime distribution with observed lifetime data. Some recent studies ([1], [3]&[4]) deal with this aspects in detail and provide the conceptual framework and methodology for such analysis. The literature dealing with complex repairable system is spares as far as Bayesian inference is concerned.

In this paper we consider a 1-out – of - 2:G repairable system from Bayesian viewpoint. The classical inference of the above mentioned repairable system was studied by Sarmah and Dharmadhikari [5]. The Bayesian inference of the above mentioned repairable system with different prior distributions were studied by Dey and Sarmah [2]. The object of the present paper is to obtain Bayesian estimates of the parameter $\theta$ using two prior distributions under three loss functions. Based on a Monte Carlo Study, the RMSE are compared with those of their Bayes’ estimators under above mentioned prior distributions and loss functions.

2. Model Description:

The system is characterized by the following set of assumptions:
(a). The system consists of two identical units and a repair facility.
(b). Initially, one unit starts operating and it goes for repair as soon as it fails.
(c). The repair discipline is FCFS and repairs are perfect.
(d). Switchover time is assumed to be negligible.
(e). The failure time distribution of the on-line unit and the repair time distribution of units under repair are assumed to be independent exponential having the same parameter $\theta$. 

3. Inspection Policy:
The system is observed under an inspection policy that, inspection is made at the completion of a repair provided that it starts at the beginning of a repair. This leads us to a situation where separate observations on the unit performance and on repair facility are not feasible. As a consequence, available records are the number of failures that occurred in the time interval between two consecutive repair epochs.

4. Stochastic Process and Sampling Scheme:
Let \( X(t) = i \), be the number of non-operative units in the system at time \( t \); \( i = 0,1,2 \).
\[ \therefore \{X(t); t \geq 0\} \text{ is a stochastic process with state space } S = \{0,1,2\}. \]
To carry out the estimation procedure for the parameter \( \theta \) we take a random sample under a specified sampling scheme which is mentioned below.
Let the available records be the epochs at which the system visits state 1. Therefore the stochastic process under the above scheme is a renewal process with inter-arrival time distribution.

\[ h(t) = 2\theta \exp(-\theta t)(1 - \exp(-\theta t)) \]

Let \( T_k \) be the time epoch at which the repair facility takes up a unit for a repair for the \( (k+1) \)th time.

\[ T_k = T_{k+1} - T_k ; k=1,2, . . . , n \]
be the length of interval between \((k+1)^{th}\) and \(k^{th}\) repair epochs, then \( T_k = \max(X_k, Y_{k-1}); k \geq 1 \), where \( X_k \) is the life time of a unit which is taken up for operation as an effect of \( k^{th} \) switch over and \( Y_k \) is the repair time of a unit when the repair facility remains busy for the \( k^{th} \) time with \( Y_0=0 \).

Clearly, \( \{ t_k \}_{k=1}^n \) constitute a random sample of size ‘n’ of a distribution having pdf:

\[ h(t) = 2\theta \exp(-\theta t)(1 - \exp(-\theta t)) \] (1)
Throughout we assume that the underlying distribution is of the form (1).

5. Estimation of Parameter:
In this section we consider the Bayes’ estimation of the parameter \( \theta \) under different loss function and different prior distributions which is mentioned below:
Here we consider two types of priors:
(a) \( g_1(\theta) \propto \frac{1}{\theta^c}, \quad c > 0 \) \hspace{1cm} (2)
(b) \( g_2(\theta) = \frac{1}{\Gamma(n)} \theta^{-n} \exp(-n\theta) ; \quad n > 0, 0 < \theta < \infty \) \hspace{1cm} (3)

Here we considered two seemingly arbitrary priors, while not attempting to represent the actual prior knowledge about the parameter \( \theta \).
Now for the priors (2) and (3), the respective posteriors of \( \theta \) are:
\[ f_1(\theta|t) = \frac{\theta^{n-c} \left[ \exp(-n\theta t) - \exp(-2n\theta t) \right]}{\Gamma(n-c+1)} - \frac{\Gamma(n-c+1)}{(nt)^{n-c+1}} \]

\[ f_2(\theta|t) = \frac{\theta^{2n-1} \left[ \exp(-\theta(1+nt)) - \exp(-\theta(1+n\tilde{t})) \right]}{\Gamma(2n)} - \frac{\Gamma(2n)}{(1+nt)^{2n} - (1+2nt)^{2n}} \]

6. Loss Functions and Corresponding Estimates:

We consider three different loss functions:

(i) Squared-error loss function:
\[ L_1(\theta^*, \theta) = (\theta^* - \theta)^2 \]

(ii) El-Sayyad’s loss function:
\[ L_2(\theta^*, \theta) = \theta^1 (\theta^* - \theta^1)^2 \]

where \( \theta^* \) is an estimator of \( \theta \).

(iii) Linex loss function:
\[ L_3(\theta^*, \theta) = b \{ e^{-a(\theta^* - \theta)} - a (\theta^* - \theta) - 1 \} \]

where \( b > 0, a \neq 0 \).

With squared error loss function, Bayes estimators for \( \theta \) with posterior densities (4) and (5) come out as:

\[ \theta_1^* = \frac{\Gamma(n-c+2)}{(nt)^{n-c+2}} - \frac{\Gamma(n-c+2)}{(2nt)^{n-c+2}} \]

\[ \theta_2^* = \frac{2n+1}{1+n\tilde{t}} \left[ \frac{1 - (\frac{1+n\tilde{t}}{1+2nt})^{2n+1}}{1 - (\frac{1+n\tilde{t}}{1+2nt})^{2n}} \right] \]

With El-Sayyad loss function, the corresponding Bayes’ estimators for \( \theta \) with posterior distribution (4) and (5) come out as:
\[ \theta_3^* = \left[ \frac{\Gamma(n-c+l+r+1)}{(nt)^{n-c+l+r+1}} - \frac{\Gamma(n-c+l+1)}{(nt)^{n-c+l+1}} \right]^{1/r} \]

\[ \theta_4^* = \left[ \frac{\Gamma(2n+l+r)}{(1+nt)^{2n+l+r}} - \frac{\Gamma(2n+l)}{(1+nt)^{2n+l}} \right]^{1/r} \]

With Linex loss function, the corresponding Bayes’ estimators for \( \theta \) with posterior distributions (4) and (5) come out as:

\[ \theta_5^* = -\frac{1}{a} \log a \left[ \frac{1}{(nt+a)^{n-c+1}} - \frac{1}{(2nt+a)^{n-c+1}} \right] \]

\[ \theta_6^* = -\frac{1}{a} \log a \left[ \frac{1}{(1+nt+a)^{2n}} - \frac{1}{(1+2nt+a)^{2n}} \right] \]

Using the Bayes’ estimators \( \theta_1^*, \theta_2^*, \theta_3^*, \theta_4^*, \theta_5^*, \theta_6^* \), quasi- Bayes estimators of reliability are obtained as:

\[ R_1(t) = 2e^{-\theta_1^* t} - e^{-2\theta_1^* t} \]

\[ R_2(t) = 2e^{-\theta_2^* t} - e^{-2\theta_2^* t} \]

\[ R_3(t) = 2e^{-\theta_3^* t} - e^{-2\theta_3^* t} \]

\[ R_4(t) = 2e^{-\theta_4^* t} - e^{-2\theta_4^* t} \]

\[ R_5(t) = 2e^{-\theta_5^* t} - e^{-2\theta_5^* t} \]

\[ R_6(t) = 2e^{-\theta_6^* t} - e^{-2\theta_6^* t} \]
7. Simulation:
In order to compare the Bayes estimator's under different loss functions and prior distributions, we generated N=1000 samples of sizes n=10, 20, 30 from the distribution (1) with $\theta = 1$. The averages of these estimates and the corresponding mean square errors (rmse) were computed. We report the result in the following Tables. The entries within parenthesis indicate the rmse.

If the rmse is accepted as an index of precision, it appears from Table-1 that Bayes’ estimates of the parameter $\theta$ under El-Sayyad loss function with quasi-prior gives better results than others. In case of reliability, it appears from Table-2 that Quasi Bayes’ estimates of reliability under Linex loss function with one parameter gamma gives better results than others.

**Table-1**: The Bayesian Estimates of the parameter $\theta_i (i=1,2,3,4,5,6)$ with different values of $c =1$ and 3, $a =1$ and -1 and $l=1$ and 2 and $r=1$ and 2

<table>
<thead>
<tr>
<th>n</th>
<th>$\theta_1$</th>
<th>$\theta_2$</th>
<th>$\theta_3$</th>
<th>$\theta_4$</th>
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<tbody>
<tr>
<td></td>
<td>c=1</td>
<td>c=3</td>
<td>c=1</td>
<td>c=3</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>.6915 (.3447)</td>
<td>.5605 (.4615)</td>
<td>.7605 (.3065)</td>
<td>.8839 (.2588)</td>
</tr>
<tr>
<td>20</td>
<td>.6765 (.3437)</td>
<td>.6058 (.4093)</td>
<td>.7190 (.3045)</td>
<td>.7787 (.2577)</td>
</tr>
<tr>
<td>30</td>
<td>.6715 (.3416)</td>
<td>.6916 (.3787)</td>
<td>.7131 (.3041)</td>
<td>.7221 (.2581)</td>
</tr>
</tbody>
</table>

**Table-1(Contd)**: The Bayesian Estimates of the parameter $\theta_i (i=5,6)$ with different values of $c =1$ and 3, $a =1$ and 2 and $l=1$ and 2 and $r=1$ and 2

<table>
<thead>
<tr>
<th>a=1</th>
<th>C=1</th>
<th>$\theta_5$</th>
<th>C=3</th>
<th>$\theta_6$</th>
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<tr>
<td></td>
<td></td>
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<td></td>
<td>a=1</td>
</tr>
<tr>
<td></td>
<td></td>
<td>a=-1</td>
<td></td>
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<tr>
<td>.6640 (.3484)</td>
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<td>.6411 (.3721)</td>
<td>1.2955 (.3290)</td>
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</table>
Table 2: The Bayesian Estimates of $R_i(t)$ $(i=1,2,3,4)$ for different values of $c=1$, $l=1,2; r=1,2; t=1$

<table>
<thead>
<tr>
<th>n</th>
<th>$R_1(t)$</th>
<th>$R_2(t)$</th>
<th>$R_3(t)$ $l=1,r=1$</th>
<th>$R_4(t)$ $l=1,r=1$</th>
<th>$R_3(t)$ $l=1,r=2$</th>
<th>$R_4(t)$ $l=1,r=2$</th>
<th>$R_3(t)$ $l=2,r=1$</th>
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<tr>
<td>10</td>
<td>.7423 (.1423)</td>
<td>.4391 (.1613)</td>
<td>.7075 (.1071)</td>
<td>.4391 (.1613)</td>
<td>.6900 (.0896)</td>
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<tr>
<td>20</td>
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<td>.4504 (.1500)</td>
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<td>.4504 (.1500)</td>
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Table 2(Contd): The Bayesian Estimates of $R_i(t)$ $(i=5,6)$ for different values of $c=1$, $a = -1,1; t=1$

<table>
<thead>
<tr>
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<th>$R_5(t)$ $a=-1$</th>
<th>$R_6(t)$ $a=1$</th>
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<td>.7436 (.1432)</td>
<td>.4817 (.1187)</td>
<td>.4652 (.1352)</td>
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<tr>
<td>20</td>
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<tr>
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<td>.7587 (.1583)</td>
<td>.4708 (.1295)</td>
<td>.4625 (.1379)</td>
</tr>
</tbody>
</table>

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References: