Evaluation of $\bar{X}$ and S charts when Standards Vary Randomly

***Aamir Saghir**

Department of Statistics, UAJK Muzaffrabad, Pakistan.

E-mail: aamirstat@yahoo.com.

Abstract

The study proposes control limits for $\bar{X}$ and S charts using Bayesian framework assuming the normality of the quality characteristic. The study deals with the analysis of the robust character of charts for variables when variations in lot-to-lot quality are suspected. Our approach consists of two stages, (i) construction of the control limits based on Bayesian framework (ii) evaluation of the proposed control limits using sampling and Bayesian Inference. Evaluation of the proposed control limits is examined using the power curve obtained for different hypothetical specifications of the data generating process. The proposed control limits based on predictive distribution are more efficient than the usual control limits in detecting a shift in parameter of the process.

Key Words: Control charts; Bayesian Analysis; Predictive distribution; Power Curve

1 Introduction

Control charts are Statistical process control tools that are widely used to monitor a process. In the literature there are two types of control charts (i) variable control charts and (ii) attributes control charts. This study deals only with variable charts. Walter Shewhart variable Location charts monitor the process mean while dispersion charts monitor the process variability. These control charts are used to detect unfavorable changes in the process with respect to some quality characteristics. The changes may occur both in the average as well in the variability of the relevant quality characteristics. A fundamental assumption in the development of charts for variables is that the underlying distribution of the quality characteristics is normal. Here we assuming that the lot-to-lot quality (process standard) observed after a fixed time interval remains constant throughout. The constant environmental stress on the operating conditions of the process over a long period leads to be unduly restrictive and unrealistic assumption about the constant standards of the process. The situation becomes alarming when one is going for quality control of the process of the same nature accomplishing the same task in varying conditions. Obviously, for overcoming the situation, it seems logical to assume variations in process standard represented by known suitable prior distribution. More so, the process control (P.C) is continuous quality valuation process and, as such, in all P.C. techniques, a strong prior information representing variations in quality is available.

Biswas (1997) and Montgomery (1991) put the sampling inspection plans for attributes using prior information about process standards in the Bayesian framework. Brush (1986) made a comparison between Bayes acceptance sampling plans and classical sampling plans. Sharma and Bhutani (1992) extended the concept of modified classical consumer's risk and Bayes consumer's risk. In all these studies the main emphasis has been to update the prior distribution with experimental data.

In this study we proposed Bayesian control limits for mean chart namely $\bar{X}$-Chart and for standard deviation namely S-Chart based on predictive distribution approach following the pioneering work of Guttman (1959), Aitchison (1964), Lawless (1975), Bain (1978), Engelhardt and Bain (1978), Chhikara and Guttman (1982), Menzefricke (2002), and Chen et.al (2005). The power curve has been developed as a performance measure of the proposed control limits following Scheffe (1949), Duncan (1951), Nelson (1985), Riaz and Saghir (2007). The power curve of the proposed control limits has been compared with those of well known control limits of $\bar{X}$ and S Charts following Tuprah and Ncube (1987), Acosta-Mejia et al. (1999), Ding et al. (2005), Riaz and Saghir (2007).

2 Proposed Control Limits

This section gives a general outline of the proposed approach to control chart based on Bayesian framework assuming the parameter uncertainty. The construction and evaluation of proposed control limits is discussed in the following two subsections.

2.1 Construction of Control Limits

Let $x_1, x_2, ..., x_n$ be a random sample of size n is drawn from Normal Distribution with unknown mean μ and known variance σ^2. The probability density function and the likelihood function of random variable X are defined respectively as:

$$f(X|\mu,\sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(x_i-\mu)^2} \quad ; \qquad -\infty < x_i < \infty,\ \sigma > 0 \qquad (2.1.1)$$

and

$$L(X|\mu,\sigma^2) = \prod f(x_i|\mu,\sigma^2) \tag{2.1.2}$$

where $f(x_i|\mu,\sigma^2)$ is the probability density function of the random variable X. The parameter μ has simple family of conjugate prior distribution (Normal distribution). The usual prior used for unknown mean (Menzefricke, 2002) is normal distribution with p.d.f:

$$p(\mu,\theta_1,\theta_2) = \frac{1}{\theta_2\sqrt{2\pi}} e^{-\frac{1}{2\theta_2}(\mu-\theta_1)^2} \quad ; \quad -\infty < x_i, \theta_1 < \infty,\ \theta_2 > 0 \tag{2.1.3}$$

where $\{\theta_1, \theta_2\}$ are hyper-parameters of the prior distributions as defined in Menzefricke (2002) and similar to hyper-parameter $\{1, \sigma_x^{-2}\}$ used for unknown precision, r, of Gaussian mixture model in Chen et al.(2005). Here we assume that the standard deviation σ is assumed to be known.

Then the posterior distribution of μ for given $x_1, x_2, ..., x_n$ will be:

$$p(\mu|x) = \frac{1}{\tau^2\sqrt{2\pi}} e^{-\frac{1}{2\tau^2}(\mu-\lambda)^2} \quad ; \quad -\infty < \mu, \lambda < \infty, \tau^2 > 0 \tag{2.1.4}$$

Finally, in view of (2.1.1) and (2.1.4), the predictive distribution of the future observations or updated compound distribution of Y following Chhikara and Guttman (1982), Sinha (1986), Menzefricke (2002), Chen et al. (2005) can also be obtained as:

$$f(Y|x) = \int_{-\infty}^{\infty} f(X,\mu,\sigma^2)\, p(\mu|x)\, d\mu$$

$$= \frac{1}{\beta\sqrt{2\pi}} e^{-\frac{1}{2\beta^2}(Y-\lambda)^2} \ ; \ -\infty < Y, \lambda < \infty, \beta^2 > 0 \tag{2.1.5}$$

The distribution of Y in (2.1.5) is normal with mean $E(Y) = \lambda$ and variance $V(Y) = \beta^2 = \sigma^2 + \tau^2$.

The central line, lower control limit and upper control limit are the three parameters of any Shewhart type control chart. Assuming, the measurable quality characteristic, X, to be normally distributed with standards μ and σ, the usual control limits for $\overline{X}$ -chart are:

$$\left.\begin{aligned} UCL_{\overline{X}} &= \mu' + A\sigma' \\ CL_{\overline{X}} &= \mu' \\ LCL_{\overline{X}} &= \mu' - A\sigma' \end{aligned}\right] \quad (2.1.6)$$

and for standard deviation chart namely S -Chart are:

$$\left.\begin{aligned} UCL_S &= B_6\sigma' \\ CL_S &= \sigma' \\ LCL_S &= B_5\sigma' \end{aligned}\right] \quad (2.1.7)$$

where μ' and σ' are specified values for μ and σ.

When μ and σ both are unknown, the control limits for $\overline{X}$ and S - Charts are:

$$\left.\begin{aligned} UCL_{\overline{X}} &= \overline{\overline{X}} + A_2\overline{S} \\ CL_{\overline{X}} &= \overline{\overline{X}} \\ LCL_{\overline{X}} &= \overline{\overline{X}} - A_2\overline{S} \end{aligned}\right] \quad (2.1.8)$$

and for standard deviation chart namely S -Chart are:

$$\left.\begin{aligned} UCL_S &= B_4\overline{S} \\ CL_S &= \overline{S} \\ LCL_S &= B_3\overline{S} \end{aligned}\right] \quad (2.1.9)$$

where, $\overline{\overline{X}}$ and $\overline{S}$ are respectively the averages of means and standard deviations of k - subgroups each of size n. The control factors A, A_2, B_3, B_4, B_5 and B_6 have been tabulated for various values of n and given in Montgomery (1991).

Now in the situation where variations in process mean are assumed to be represented by the prior distribution of μ in (2.1.3) and using the sample information variation in the process mean can be updated in the form of posterior distribution in (2.1.4). Using this posterior distribution of μ, the basic distribution of X in (2.1.1) is further updated and the control limits for $\overline{X}$ and S charts become:

$$\left.\begin{aligned} UCL_{\overline{X}} &= \lambda' + A\beta' \\ CL_{\overline{X}} &= \lambda' \\ LCL_{\overline{X}} &= \lambda' - A\beta' \end{aligned}\right] \quad (2.1.10)$$

and

$$\left.\begin{aligned} UCL_S &= B_6\beta' \\ CL_S &= \beta' \\ LCL_S &= B_5\beta' \end{aligned}\right] \qquad (2.1.11)$$

Here, λ' and β' are specified values of λ and β respectively.

2.2 Evaluation of Control Limits

Let us now evaluate the rejection region when the stable process model has changed and the future data are generated by a different model, that is,

$$f(y\,|\,\mu_0, \sigma^2, a) = N(\mu_0 + a_1 = \mu', a_2^2\sigma^2) \qquad (2.2.1)$$

Following the pioneering work of Shewhart, first we have to control the variation of the process then we investigate the performance of the mean control chart namely $\overline{X}$-Chart limits. The power curves in the corresponding situations provide a measure of the sensitivity of the control charts, that is, their ability to detect a shift in the mean of the process quality. The power function of control limits for S-Chart in the hypothetical situation of parameter in usual approach is defined as:

$$B\left(S, a_2^2\sigma^2\right) = 1 - P[LCL_S \le S \le UCL_S\,|\,a_2\sigma]$$

$$= 1 + \phi(C_1) - \phi(C_2) \qquad (2.2.2)$$

Here $S \sim N\{c_4(a_2\sigma), (1-c_4^2)(a_2^2\sigma^2)\}$, $C_1 = \dfrac{B_5\sigma' - c_4(a_2\sigma)}{a_2\sigma\sqrt{1-c_4^2}}$, $C_2 = \dfrac{B_6\sigma' - c_4(a_2\sigma)}{a_2\sigma\sqrt{1-c_4^2}}$, σ' is a specified value of σ and a_2 is an amount of shift in the value of σ.

Similarly the power function of control limits for $\overline{X}$-Chart in the hypothetical situation of parameter in usual approach is defined as:

$$\alpha\left(\overline{X}, \mu'\right) = 1 - P[LCL_{\overline{X}} \le \overline{X} \le UCL_{\overline{X}}\,|\,\mu']$$

$$= 1 - \phi\left(\frac{UCL_{\overline{X}} - \mu'}{\sigma/\sqrt{n}}\right) + \phi\left(\frac{LCL_{\overline{X}} - \mu'}{\sigma/\sqrt{n}}\right) \qquad (2.2.3)$$

where $\phi(t)$ stands for the cumulative distribution of a standard normal distribution. We are thus examining the performance of the rejection region in the hypothetical situation when the population mean is known to μ' and the control process variation at σ.

Therefore, the corresponding power function of control limits for S-Chart in the hypothetical situation of parameter in Bayesian approach is defined as:

$$B'\left(S, a_2^2\beta^2\right) = 1 - P[LCL_S \leq S \leq UCL_S | a_2\beta]$$

$$= 1 + \phi(C_1) - \phi(C_2) \qquad (2.2.4)$$

Here $S \sim N\{c_4(a_2\beta), (1-c_4^2)(a_2^2\beta^2)\}$, $C_1 = \dfrac{B_5\beta' - c_4(a_2\beta)}{a_2\beta\sqrt{1-c_4^2}}$, $C_2 = \dfrac{B_6\beta' - c_4(a_2\beta)}{a_2\beta\sqrt{1-c_4^2}}$, β' is a specified value of β and a_2 is an amount of shift in the value of β.

Similarly the power function of control limits for $\overline{X}$ -Chart in the hypothetical situation of parameter in usual approach is defined as:

$$\alpha'\left(\overline{X}, a_1 + \lambda = \lambda'\right) = 1 - P[LCL_{\overline{X}} \leq \overline{X} \leq UCL_{\overline{X}} | \lambda']$$

$$= 1 - \phi\left(\frac{UCL_{\overline{X}} - \lambda'}{\beta/\sqrt{n}}\right) + \phi\left(\frac{LCL_{\overline{X}} - \lambda'}{\beta/\sqrt{n}}\right) \qquad (2.2.5)$$

We are thus examining the performance of the rejection region in the hypothetical situation when the population mean is known to λ' and the control process variation at β.

3 Comparison

We made a comparison between usual and Bayesian approaches based limits for $\overline{X}$ and S -charts. For this purposes we consider the initial distribution of X in (2.1.1) be $N(400, 64)$, and that for μ in (2.1.2) be $N(400,17)$. Further suppose the process sample information of size n =9 yields $\overline{X}$ =403 and S_x^2 =65.5906. Then,

$$\lambda = \frac{n\overline{X}\theta_2 + \theta_1\sigma^2}{\sigma^2 + n\theta_2^2} = 402.12$$

$$\beta^2 = \frac{\sigma^4 + (n+1)\sigma^2\theta_2}{\sigma^2 + n\theta_2} = 69.01$$

The control limits based on Bayesian and usual approaches for $\overline{X}$ and S charts has been calculated using the above data and for comparison purposes are provided in the following table 1.

Table 1: Control Limits for $\overline{X}$ and S -Charts

Control Limits	**$\overline{X}$ -Chart**	**S -Chart**
Usual Control Limits	$\overline{X} \pm A\sigma$ or (392, 408)	$B_5\sigma = 1.856$

		$B_6\sigma = 13.656$
Bayesian Control Limits	$\lambda \pm A\beta$ or (393.81, 410.43)	$B_5\beta = 1.927$ $B_6\beta = 14.180$

From Table.1 it is clear that the control limits of $\overline{X}$ and S charts obtained based on predictive distribution, which formally incorporate the parameter uncertainty, are wider than the usual control limits, which ignores this uncertainty and assume $\overline{X} = \mu$ and $S^2 = \sigma^2$ as discuss by Menzefricke (2002).

3.1 Power and ARL Comparison for S-Chart

First we investigate the performance of S-Chart in the situation when shift in the variation of the process has occurred. Using the expressions for power functions in (2.2.2) and (2.2.4), the points on the power curves for varying a_2 have been tabulated in table 3.1.1 and power curve is provided in figure 3.1.1 for S-Chart. Also the ARL functions for both approaches have been tabulated in table 3.1.1 and in figure 3.1.2.

Table 3.1.1: Power and ARL functions for S-Chart

a_2	$B(a_2\sigma)$	$ARL_{a_2\sigma}$	$B'(a_2\beta)$	$ARL_{a_2\beta}$
1	0.0027	374.2747	0.0027	374.2747
2	0.6813	1.4677	0.6816	1.4671
3	0.9483	1.0545	0.9486	1.0542
4	0.9864	1.0138	0.9865	1.0137
5	0.9947	1.0053	0.9948	1.0052
6	0.9974	1.0026	0.9974	1.0026
8	0.9990	1.0010	0.9990	1.0010
10	0.9995	1.0005	0.9995	1.0005
15	0.9998	1.0002	0.9998	1.0002

Fig.3.1.1: Power Curve for Usual and Bayesian Limits of *S*-Chart

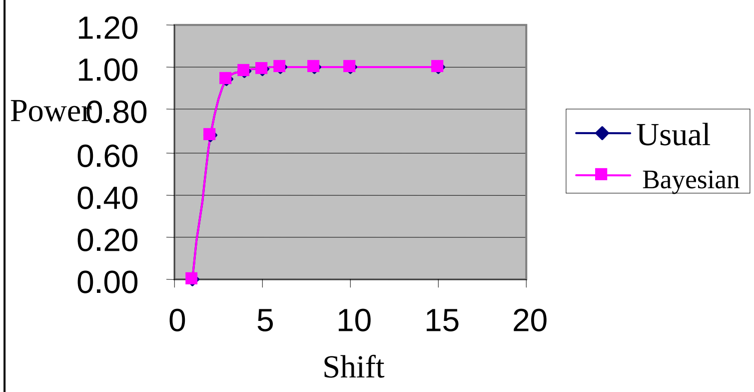
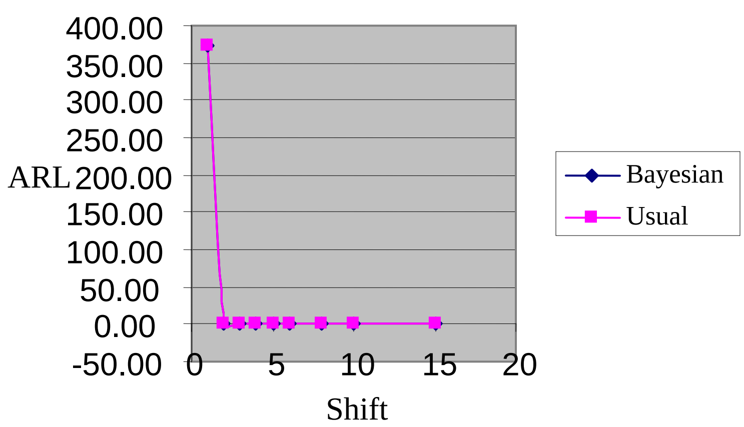


Fig.3.1.2: ARL curve for Usual and Bayesian Limits of $S-$Chart



The power of detecting a shift in the process standard deviation is more for the Bayesian approach based limits as compare to usual approach based limits as it is clear from the table 3.1.1 and fig. 3.1.1. Also the ARL for the Bayesian limits is smaller compare to usual limits as it is obvious from table 3.1.1 and fig.3.1.2.

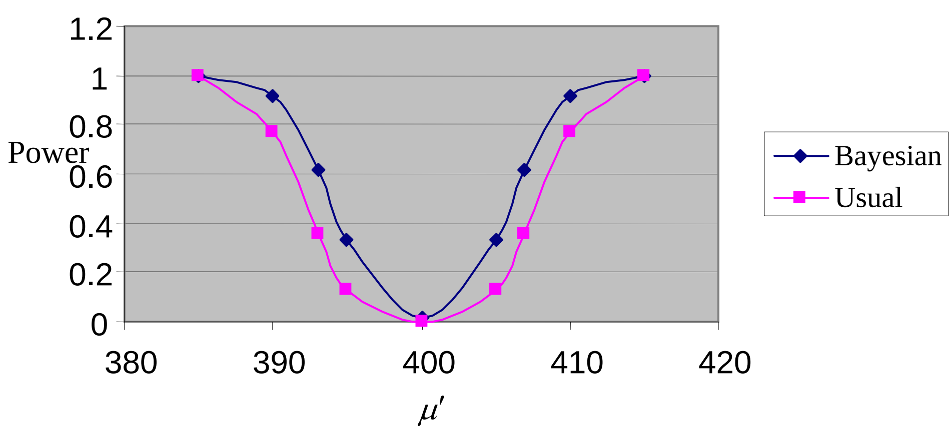
3.2 Power and ARL Comparison for $\overline{X}$ -Chart

After investigation of the performance of S-Chart, we now investigate the performance of the $\overline{X}$-Chart for the hypothetical values of parameter μ and λ for sampling and predictive distribution respectively. Using the expressions for power functions in (2.2.3) and (2.2.5), the points on the power curves for varying $\mu' = \lambda'$ (for statistically validity the reference point is assumed to be equal) have been tabulated in table. 3.2.1 and power curve is provided in figure 3.2.1 for $\overline{X}$ -Chart. Also the ARL functions for both approaches have been tabulated in table 3.2.1.

Table 3.2.1: Power and ARL functions for $\overline{X}$ -Chart

μ'	$\alpha\ (a_1\mu)$	$ARL_{a_1\mu}$	$\alpha'(a_1\lambda)$	$ARL_{a_1\lambda}$
385	0.995	1.005025	0.999	1.001001
390	0.774	1.29199	0.915	1.092896
393	0.356	2.808989	0.614	1.628664
395	0.129	7.751938	0.334	2.994012
400	0.003	333.3333	0.013	76.92308
405	0.129	7.751938	0.334	2.994012
407	0.356	2.808989	0.614	1.628664
410	0.774	1.29199	0.915	1.092896
415	0.995	1.005025	0.999	1.001001
385	0.995	1.005025	0.999	1.001001
390	0.774	1.29199	0.915	1.092896
393	0.356	2.808989	0.614	1.628664
395	0.129	7.751938	0.334	2.994012

Fig. 3.2.1: Power curve for usual and Bayesian Limits of $\overline{X}$ -Chart



From table 2.2.1 and figure 3.2.1 it's obvious that Bayesian approach is more efficient than the usual approach in detecting a shift in the mean level of the continuous process when process standards vary randomly.

4 Conclusion

The analysis clearly indicates that $\overline{X}$ and S chart are non-robust in respect to variation in process mean as well as in process standard deviation and the same limits be used when variation in the process standards are expected. In the situation, when prior variation is expected in the process standards randomly, the control limits based on Bayesian framework are more non-robust than the limits obtained by usual approach. The power of detecting the shifts in the parameter of the continuous process is more for Bayesian inference control limits than the usual inference control limits as obvious from the tables 3.1.1-3.2.1 and figures 3.1.1-3.2.1. Thus the Bayesian approach has a definite advantage over the frequentist approach as discussed by Chhikara and Guttman (1982) and in the article.

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Aamir Saghir,

Lecturer Department of Statistics,

University of AJ&K, Muzaffarabad,

AJ& K, Pakistan.