Discrete Maxwell Distribution

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Abstract: *Maxwell distribution plays an important role in Physics and other allied sciences. This paper introduces a discrete analogue of the Maxwell Distribution, called Discrete Maxwell Distribution or dMax Distribution. This distribution is suggested as a suitable reliability model to fit a range of discrete lifetime data. Distributional properties, reliability characteristics and its relationships with continuous Maxwell distribution are also elicitated. The article displays a simulation study on fitting dMax distribution as compared to other popular discrete models. Maximum likelihood estimation and Bayesian analysis for the dMax distribution are also explored.*

Keywords: Maxwell distribution, discrete lifetime models, reliability, failure rate, Maximum Likelihood Estimation, Bayes estimation.

2000 Mathematical Subject Classifications: Primary: 62E20, 62F10, 62N05; Secondary: 65C20, 68U20.

1. Introduction

In reliability theory many continuous lifetime models have been suggested and studied. See for example, Lawless (1982), Sinha (1986). However, it is sometimes impossible or inconvenient to measure the life length of a device, on a continuous scale. In practice, we come across situations, where lifetime of a device is considered to be a discrete random variable (rv). For example, in an on/off-switching device, the lifetime of the switch is a discrete rv. Also, the number of voltage fluctuations which an electrical or electronic item can withstand before its failure, is a discrete rv.

If the lifetimes of individuals in some population are grouped or when lifetime refers to an integral number of cycles of some sort, it may be desirable to treat it as a discrete rv. When a discrete model is used with lifetime data, it is usually a multinomial distribution, which arises because effectively continuous data have been grouped. Some situations may demand for another discrete distribution, usually over the non-negative integers. Such situations are best treated individually, but generally one tries to adopt one of the standard discrete distributions.

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In the last two decades, standard discrete distributions like geometric and negative binomial have being employed to model lifetime data. However, there is a need to find more plausible discrete lifetime distributions to fit to various types of lifetime data. For this purpose, popular continuous lifetime distributions can be helpful in the following manner.

Discretizing a continuous distribution:

A continuous failure time model can be used to generate a discrete model by introducing a grouping on the time axis. If the underlying continuous failure time X has the Survival function (Sf) $S(x) = P[X \geq x]$ and times are grouped into unit intervals so that the discrete observed variable is $dX = [X]$, the largest integer part of X, the probability mass function (pmf) of dX can be written as

$$p(x) = P[dX = x] = P[x \leq dX < x+1] = S(x) - S(x+1);\ x = 0,1,2\ldots \quad (1)$$

The pmf of rv dX can be viewed as discrete concentration of the probability density function (pdf) of X. The first and easiest in this approach is the geometric distribution with pmf

$$p(x) = \theta^{x} - \theta^{x+1};\ x = 0,1,2\ldots$$

which is obtained by discretizing the exponential distribution with Sf

$S(x) = e^{-\lambda x};\ \lambda, x > 0.$ Here $\theta = e^{-\lambda}$, $(0 < \theta < 1)$.

Following this approach, Nakagawa (1975) discretized the Weibull distribution. Stein and Dattero (1984) discussed a new discrete Weibull distribution. Dilip Roy (2003, 2004) considered discrete normal and Rayleigh distributions. These authors have also studied the distributional and reliability properties of such discritized distributions.

The Maxwell distribution was first introduced in the literature by J.C. Maxwell (1860) and again described by Boltzman (1870) with a few assumptions. This distribution defines the speed of molecules in thermal equilibrium under some conditions as defined in statistical mechanics. For example, this distribution explains many fundamental gas properties in kinetic theory of gases; distribution of energies and moments etc. Seeing the usefulness of Maxwell distribution, one of the leading softwares 'Mathematica' has included Maxwell distribution and its properties in its software library.

Tyagi and Bhattacharya (1989 a&b) considered Maxwell distribution as a lifetime model for the first time. They obtained the minimum variance unbiased estimator

(MVUE) and Bayes estimator of the parameter and reliability function of this distribution. Chaturvedi and Uma Rani (1998) generalized Maxwell distribution by introducing one more parameter and obtained classical and Bayesian estimation procedures for it.

In this paper we first review the Maxwell distribution as a lifetime model. Then, we derive the dMax distribution with its important distributional properties and reliability characteristics. The equivalence of continuous and discrete Maxwell distributions has been established. We also, obtain the maximum likelihood estimates (MLE) and Bayes estimates of the parameter and reliability characteristics. A simulation study is performed at the end, which involves expected values and risks of the estimates. The fitting of dMax distribution has been shown to be better than many other popular discrete distributions.

2. The model

2.1 Continuous Maxwell distribution

A lifetime rv X follows the Maxwell (or Maxwell-Boltzman) distribution MW(θ) if its pdf is given by

$$f(x) = \frac{4}{\sqrt{\pi}}\frac{1}{\theta^{3/2}}\, x^2\, e^{-x^2/\theta}; \qquad x, \theta > 0 \tag{2}$$

The cumulative distribution function (cdf) is

$$F(x) = \frac{1}{\Gamma(3/2)}\Gamma\left(\frac{3}{2}, \frac{x^2}{\theta}\right); \quad x, \theta > 0$$

where $\Gamma(a,x) = \int_0^x u^{a-1}e^{-u}du$ is the incomplete gamma function.

The r^{th} moment is given by,

$$E\left(X^r\right) = \frac{2}{\sqrt{\pi}}\theta^{r/2}\Gamma\left(\frac{r+3}{2}\right); \; r > -3.$$

In particular, E(X) $= 2\sqrt{\frac{\theta}{\pi}}$; V(X) $= \frac{\theta}{2\pi}(3\pi - 8) = 0.2268\,\theta$ and coefficient of variation$= \frac{\sqrt{(0.2268\pi)}}{2} = 0.4220$. Also, Pearson's coefficient of skewness and kurtosis respectively, are $\gamma_1 = \sqrt{\beta_1} = \frac{16 - 5\pi}{\left(\frac{3}{2}\pi - 4\right)^{3/2}} = 0.4857$ and $\gamma_2 = \beta_2 - 3 = \frac{-4\left(3\pi^2 - 40\pi + 96\right)}{\left(3\pi - 8\right)^2} =$

0.1081. Thus, Maxwell distribution is positively skewed and leptokurtic distribution with Mode = $+\sqrt{\theta}$ and Median = $\sqrt{\frac{\theta}{\pi}}\left(\frac{\sqrt{\pi}+4}{3}\right)$ = 1.0856 $\sqrt{\theta}$. The Sf is written as,

$$S(x) = \frac{4}{\sqrt{\pi}}\frac{1}{\theta^{3/2}}\ J(x,2,\theta); \qquad x,\theta > 0 \tag{3}$$

where $\mathrm{J}(x,k,\theta) = \int_x^{\infty} u^k e^{-u^2/\theta} du$

The failure rate (FR) is given by,

$$r(x) = \frac{f(x)}{S(x)} = \frac{x^2 e^{-x^2/\theta}}{\mathrm{J}(x,2,\theta)};\ x > 0$$

Theorem 1: The Maxwell distribution is increasing failure rate (IFR) distribution.

Proof: By lemma 5.9,(p 77, Barlow and Proschan (1975)), for this it is sufficient to show that log of the pdf is concave. Let $\phi(x) = \log f(x)$. Now, $\phi(x)$ is defined and twice differentiable in $(0,\infty)$. Also, second derivative of $\phi(x)$ w.r.t. x is $\left(-\frac{2}{\theta}-\frac{2}{x^2}\right) < 0, \forall x > 0$.

So, $\phi(x) = \log f(x)$ is concave in x.

Hence the theorem.

2.2 Discrete Maxwell distribution

Using the discretization approach given in equation (1) and the Sf of continuous Maxwell distribution, we define the discrete Maxwell distribution dMax (θ) as,

$$p(x) = \frac{4}{\sqrt{\pi}}\frac{1}{\theta}Q(x,2,\theta);\ x=0, 1, 2\ldots,\ \theta>0$$

where $Q(x,k,\theta) = \int_x^{x+1} u^k e^{-u^2/\theta} du$ = J(x, k, θ) - J(x +1, k, θ).

The plots of pmf of dMax (θ), given in Figure 1 show that dMax distribution is positively skewed and approaches to normality for large θ.

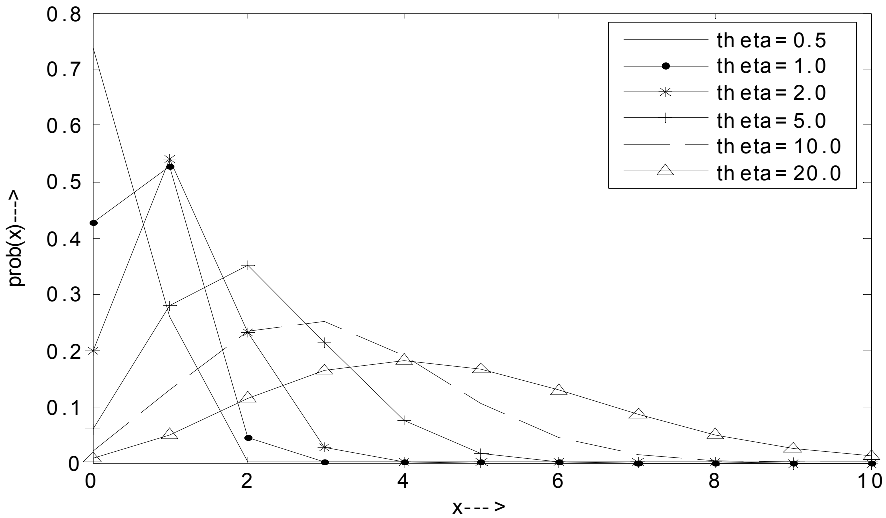

Fig.1: pmf plot for dMax(θ)

The r^th^ moment is

$$\mu_r' = E\left(X^r\right) = \frac{4}{\sqrt{\pi}}\frac{1}{\theta^{3/2}}\sum_{x=0}^{\infty} x^r Q\left(x,2,\theta\right);\ r=1,2,3\ldots;\ \theta>0$$

Theorem 2: Let X be a non-negative continuous rv with $E\left(X^r\right)<\infty, \forall r=1,2,3\ldots$. Then, $E\left(\left[X\right]^r\right)<\infty$.

Proof: Proof is straight forward, since $0 \le \left[X\right] \le X$, almost surely (a.s.).

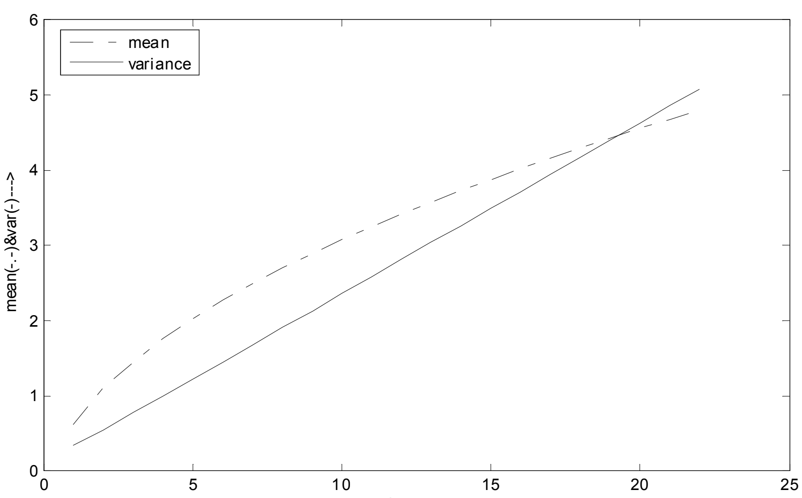

Fig. 2: Plot for mean and variance of dMax(θ)

Remark: By above theorem, mean and variance of dMax(θ) distribution are finite. Figure 2 displays that for θ<19.264774, mean < variance.

Reliability characteristics of dMax(θ):

The Sf of dMax(θ) as well as of MW(θ) at integer points of x will be equal to

$$S(x) = \frac{4}{\sqrt{\pi}}\frac{1}{\theta^{3/2}}\,J(x,2,\theta); \qquad x = 0,1,2....,\theta > 0$$

The FR of dMax(θ) is given by,

$$r(x) = \frac{p(x)}{S(x)} = \frac{Q(x,2,\theta)}{J(x,2,\theta)}$$

$$= 1 - \frac{J(x+1,2,\theta)}{J(x,2,\theta)}; \quad x=0,1,2...;\theta > 0$$

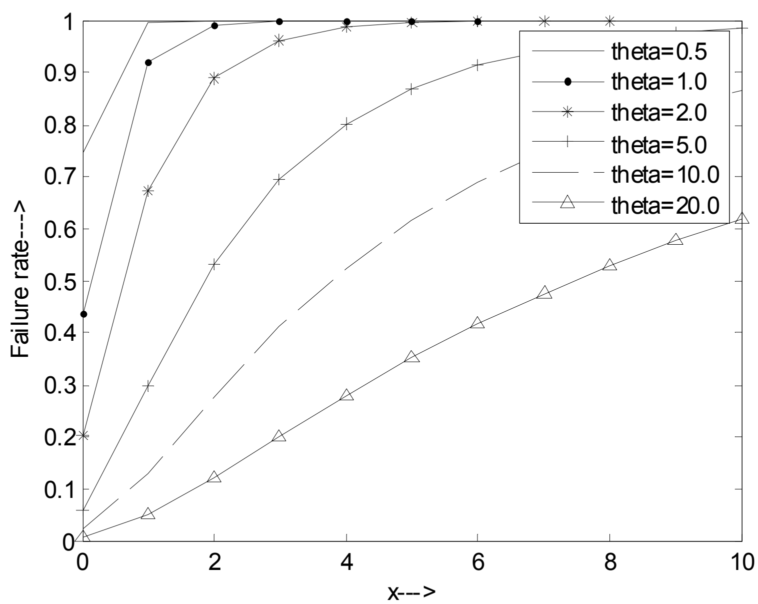

Fig. 3: Plot for Failure Rate of dMax(θ)

Lemma 1: If Z is a continuous rv with increasing (decreasing) failure rate IFR (DFR) distribution, then $Z_d = [Z]$ is discrete increasing (decreasing) failure rate dIFR (dDFR).

Proof: (cf. Roy and Dasgupta (2001)).

Theorem 3: The dMax (θ) is a dIFR distribution.

Proof: Using Theorem 1 and lemma 1 proof is readily established.

For discrete distributions failure rate r(x) is a conditional probability and has unity as its upper bound. Xie et al (2002) pointed out that calling this as failure rate function might add to the confusion that is already common in industry that failure rate and failure probability are sometimes mixed-up. To solve this problem they introduced second rate of failure SRF (x) with same monotonicity as r(x). For dMax (θ) we have,

$$\text{SRF }(x) = \log\left[\frac{S(x)}{S(x+1)}\right] = \log\left[\frac{J(x,2,\theta)}{J(x+1,2,\theta)}\right].$$

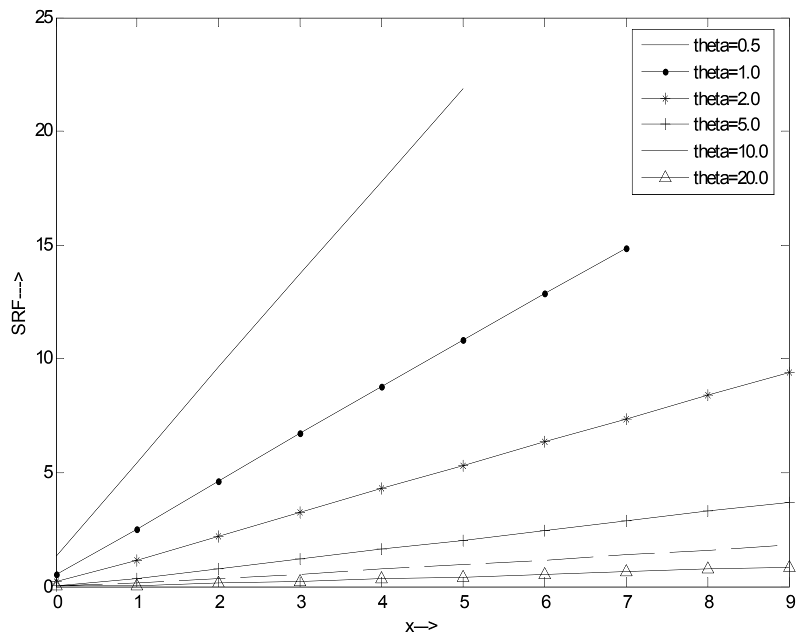

Fig. 4: Plot for Second Rate of Failure of dMax(θ)

In Figures 3 and 4, the graphs of r(x) and SRF (x) are displayed. The figures show that dMax has higher failure rates for smaller values of θ. The SRF becomes almost linear in x for all θ.

In reliability theory, classification of lifetime models is defined in terms of their Sf and other reliability characteristics. For example, the increasing (decreasing) failure rate IFR (DFR) class, increasing (decreasing) failure rate average IFRA (DFRA) class, the new better (worse) than used NBU (NWU) class, new better (worse) than used in expectation NBUE (NWUE) class and increasing (decreasing) mean residual lifetime IMRL (DMRL) class etc. The discritization of a continuous lifetime distribution retains the same functional form of the Sf, therefore, many reliability characteristics and properties shall remain unchanged. Thus, discritization of a continuous lifetime model is an interesting and simple approach to derive a discrete lifetime model corresponding to the continuous one. Also, the following chain of implications given by Kemp (2004), page 3074 for discrete distributions is very useful.

IFR / DFR $\Rightarrow$ IFRA / DFRA $\Rightarrow$ NBU / NWU $\Rightarrow$ NBUE / NWUE $\Rightarrow$ DMRL / IMRL.

Lemma 2: If X is a non-negative continuous rv and Y is a non-negative integer valued discrete rv, then $[X] \geq Y \Leftrightarrow X \geq Y$ (a.s.).

Proof: Note that,

$([X]\geq Y) \subseteq (X\geq Y) \subseteq ([X] \geq [Y]) = ([X] \geq Y)$.

where the last equality holds since Y is integer valued. Therefore, $(X\geq Y) = ([X] \geq Y)$.

Theorem 4: If X ~ MW(θ), then Y=[X] ~dMax(θ).

Proof: Consider $\forall Y = 0,1,2...$

$$\mathrm{S}_Y(y) = \mathrm{P}[\mathrm{Y} \geq \mathrm{y}] = \mathrm{P}[[\mathrm{X}] \geq \mathrm{y}]$$

$$= \mathrm{P}[\mathrm{X} \geq \mathrm{y}] = \frac{4}{\sqrt{\pi}} \frac{1}{\theta^{3/2}} \mathrm{J}(\mathrm{y},2,\theta)$$

which is Sf of dMax(θ). Hence the theorem.

Theorem 5: Let X be a non-negative continuous rv and t >0. Then, Y=[X/t] has $\mathrm{dMax}\left(\frac{\theta}{\mathrm{t}^2}\right)$ distribution for every t iff X~ MW(θ).

Proof: (if part) Let X~MW(θ). Then, $\forall x = 0,1,2...$

$$\mathrm{P}[\mathrm{Y} \geq x] = \mathrm{P}\left[\left[\frac{\mathrm{X}}{\mathrm{t}}\right] \geq x\right]$$

$$= \mathrm{P}\left[\frac{X}{t} \geq x\right] = \mathrm{P}[X \geq xt]$$

$$= \int_{xt}^{\infty} \frac{4}{\sqrt{\pi}} \frac{1}{\theta^{3/2}} e^{-w^2/\theta} w^2 dw$$

putting s=w/t $\Rightarrow$ t ds= dw

$$= \int_{x}^{\infty} \frac{4}{\sqrt{\pi}} \frac{1}{\left(\frac{\theta}{t^2}\right)^{3/2}} e^{-s^2/\left(\theta/t^2\right)} s^2 ds$$

i.e. Y~ $\text{dMax}\left(\frac{\theta}{t^2}\right)$.

(only if part) we are given that Y~$\text{dMax}\left(\frac{\theta}{t^2}\right)$ i.e. $\forall y = 0,1,2...$

$$S_Y(y) = P[Y \ge y] = P\left[\left[\frac{X}{t}\right] \ge y\right]$$

$$= P\left[\frac{X}{t} \ge y\right] = P[X \ge yt] = S_X(yt), \quad (\because \ yt \in [0,\infty)).$$

$$\Rightarrow S_X(yt) = P[Y \ge y]$$

$$= \frac{4}{\sqrt{\pi}} \frac{1}{\left(\frac{\theta}{t^2}\right)^{3/2}} \int_y^\infty e^{-s^2/(\theta/t^2)} s^2 ds$$

putting s = w/t $\Rightarrow$ ts = w $\Rightarrow$ t ds = dw.

$$\Rightarrow S_X(yt) = \frac{4}{\sqrt{\pi}} \frac{1}{\theta^{3/2}} \int_{yt}^\infty e^{-w^2/\theta} w^2 dw = \frac{4}{\sqrt{\pi}} \frac{1}{\theta^{3/2}} J(yt,2,\theta)$$

i.e. X ~ MW (θ). Hence the proof.

3. Maximum Likelihood Estimation

Let n items be put on test and their failure times are recorded as $x_1,x_2,\ldots.x_n$. If these failure times are assumed to be independently and identically distributed (iid) rv's following dMax(θ), the log-likelihood function of the sample will be,

$$\log L\,(\underset{\sim}{x}, \alpha, \theta) = n\log\left(\frac{4}{\sqrt{\pi}}\right) - \frac{3}{2} n\log\theta + \sum_{i=1}^{n} \log\ Q(x_i,2,\theta)\,; \tag{4}$$

Thus, the log-likelihood equation $\frac{\partial \log L}{\partial \theta} = 0$ becomes,

$$\frac{1}{n}\sum_{i=1}^{n} \frac{Q(x_i,4,\theta)}{Q(x_i,2,\theta)} = \frac{3}{2}\theta \tag{5}$$

The equation (5) can be solved for θ to get the MLE $\hat{\theta}$ by a suitable numerical iterative method such as Newton-Raphson (N-R) method. The seed for such iterative method can be taken as $\theta_0 = \frac{2}{3n}\sum_{i=1}^{n}(x_i + 0.5)^2$, which is an approximation to the MLE of θ in case of continuous Maxwell distribution. Also, the second derivative of the log-likelihood function is given by

$$\frac{\partial^2 \log\ L}{\partial \theta^2} = \frac{3n}{2\theta^2} + \frac{1}{\theta^4}\sum_{i=1}^{n} \frac{Q(x_i,2,\theta)\{Q(x_i,6,\theta) - 2\theta Q(x_i,4,\theta)\} - (Q(x_i,4,\theta))^2}{(Q(x_i,2,\theta))^2} \tag{6}$$

Using the general theory of MLEs, the MLE $\hat{\theta}$ follows asymptotically normal distribution $N\left(\theta, \frac{1}{I(\theta)}\right)$,where $I(\theta)$ is the Fisher's information $I(\theta) = E\left(-\frac{\partial^2 \log L}{\partial \theta^2}\right)$. This estimate of Var($\hat{\theta}$) helps us construct (1-$\alpha$)x100 % confidence interval $\hat{\theta} \pm z_{\alpha/2}\sqrt{\frac{1}{I(\hat{\theta})}}$ for θ, where $z_{\alpha/2}$ is the $(\alpha/2)^{th}$ upper point of standard normal distribution.

Also, by invariance property of MLEs, $\phi(\hat{\theta})$, a one to one function of $\hat{\theta}$, is the MLE of $\phi(\theta)$. Using this property, the MLEs of survival function, failure and second failure rates and mean lifetime (MLT) are respectively, given by,

$$\hat{S}(t) = \frac{4}{\sqrt{\pi}}\frac{1}{\hat{\theta}^{3/2}} J\left(t, 2, \hat{\theta}\right); \quad t=0, 1, 2\ldots, \theta > 0$$

$$\hat{r}(t) = 1 - \frac{J\left(t+1, 2, \hat{\theta}\right)}{J\left(t, 2, \hat{\theta}\right)};$$

$$SRF(t) = \log\left(\frac{J\left(t, 2, \hat{\theta}\right)}{J\left(t+1, 2, \hat{\theta}\right)}\right);$$

and $$M\hat{L}T = \hat{\mu} = \frac{4}{\sqrt{\pi}}\frac{1}{\hat{\theta}^{3/2}} \sum_{y=1}^{\infty} J\left(y, 2, \hat{\theta}\right).$$

4. Bayes Estimation

Let $\underset{\sim}{x}$=(x_1,x_2,…..x_n) be a random sample from dMax(θ). Here, θ is regarded as a rv having a prior family of inverted gamma distributions with pdf

$$g(\theta) = \frac{a^b}{\Gamma b}\frac{1}{\theta^{b+1}} e^{-a/\theta}; \ a, b, \ \theta > 0$$

Thus, the posterior distribution of θ given $\underset{\sim}{x}$ becomes

$$\pi\left(\theta | \underset{\sim}{x}\right) = K \frac{1}{\theta^{b'+1}} \left\{\prod_{i=1}^{n} Q\left(x_i, 2, \theta\right)\right\} e^{-a/\theta};$$

where $b' = b + (3/2)n$

and $K^{-1} = \int_0^{\infty} \frac{1}{\theta^{b'+1}} \left\{\prod_{i=1}^{n} Q\left(x_i, 2, \theta\right)\right\} e^{-a/\theta} d\theta.$

Assuming squared error loss function, as it is appropriate when decisions are gradually more damaging for larger errors, Bayes estimator θ^* of θ is the posterior mean of θ, given by,

$$\theta^* = K\int_0^\infty \frac{1}{\theta^{b'}}\left\{\prod_{i=1}^{n} Q\left(x_i, 2, \theta\right)\right\} e^{-a/\theta} d\theta .$$

Also, the Bayes estimators of other reliability characteristics are obtained as,

$$S\left(t\right)^* = K\frac{4}{\sqrt{\pi}}\int_0^\infty \frac{J\left(t,2,\theta\right)}{\theta^{b'+5/2}}\left\{\prod_{i=1}^{n} Q\left(x_i, 2, \theta\right)\right\} e^{-a/\theta} d\theta ;$$

$$r\left(t\right)^* = K\int_0^\infty \frac{Q\left(t,2,\theta\right)}{J\left(t,2,\theta\right)}\left\{\prod_{i=1}^{n} Q\left(x_i, 2, \theta\right)\right\}\frac{e^{-a/\theta}}{\theta^{b'+1}} d\theta ;$$

$$SRF\left(t\right)^* = K\int_0^\infty \log\left(\frac{J\left(t,2,\theta\right)}{J\left(t+1,2,\theta\right)}\right)\left\{\prod_{i=1}^{n} Q\left(x_i, 2, \theta\right)\right\}\frac{e^{-a/\theta}}{\theta^{b'+1}} d\theta ;$$

and $\quad MLT^* = \sum_{t=1}^{\infty} S^*\left(t\right).$

5. Simulation Study

A thorough simulation study was performed by considering various pairs of values of hyper-parameters a and b of prior distribution. Then, by taking $\theta = \frac{a}{b-1}$, the mean of the prior distribution, we generated 1000 samples of size 25 from dMax (θ). Solving likelihood equation (5) by N-R method with each of these samples, we obtained $\hat{\theta}$ along with the observed Fisher's information using equation (6). Also, Bayes estimates of θ and other reliability characteristics were computed for each sample. The results of simulation study are given in Table 1.

In the table, all entries are the means of estimates for 1000 samples of size n=25. For example, the value against $E\left(\hat{\theta}\right)$ is the mean of one thousand $\hat{\theta}$. Similarly, expected risk ER ($\hat{\theta}$) is the mean of 1000 estimates of squared deviations of $\hat{\theta}$ from θ i.e. $\left(\hat{\theta}-\theta\right)^2$.

Table 1: Simulated expectations of MLEs & Bayes estimates

(a, b) Actual θ	(2, 5) 0.5	(4, 5) 1.0	(2, 2) 2.0	(20, 5) 5.0	(50, 6) 10.0	(40, 3) 20.0
$E(\hat{\theta})$	0.4873	1.1121	2.0198	4.9202	10.3178	20.2178
$ER(\hat{\theta})$	0.0019	0.0690	0.0474	0.5691	0.7125	4.3267
$CI_{95\%}$	(0.46, 0.51)	(0.99, 1.23)	(1.77, 2.27)	(3.54, 6.30)	(4.54,19.09)	(12.26,26.43)
$E(\theta^*)$	0.5088	1.0924	2.0108	4.9218	10.2726	20.1987
$ER(\theta^*)$	0.0002	0.0518	0.0444	0.4593	0.5470	3.8896
Actual $S(2)$	0.0011	0.0460	0.2615	0.6594	0.8495	0.9402
$E(\hat{S}(2))$	0.0011	0.0716	0.2644	0.6463	0.8543	0.9401
$ER(\hat{S}(2))$	0.0001	0.0024	0.0021	0.0034	0.0002	0.0001
$E(S^*(2))$	0.0017	0.0695	0.2593	0.6400	0.8488	0.9375
$ER(S^*(2))$	0.0002	0.0019	0.0019	0.0030	0.0001	0.0001
Actual $r(2)$	0.9999	0.9904	0.8880	0.5329	0.2761	0.1221
$E(\hat{r}(2))$	0.9960	0.9804	0.8847	0.5456	0.2683	0.1222
$ER(\hat{r}(2))$	0.0001	0.0003	0.0009	0.0044	0.0005	0.0003
$E(r^*(2))$	0.9685	0.9803	0.8846	0.5504	0.2756	0.1263
$ER(r^*(2))$	0.0025	0.0003	0.0008	0.0036	0.0004	0.0003
Actual $SRF(2)$	9.6253	4.6502	2.1890	0.7611	0.3231	0.1302
$E(S\hat{R}F(2))$	9.9646	4.3824	2.1935	0.7991	.03128	0.1305
$ER(S\hat{R}F(2))$	0.9890	1.2322	0.0670	0.0225	0.0009	0.0004
$E(SRF^*(2))$	9.5835	4.5532	2.1791	0.8198	0.3247	0.1356
$ER(SRF^*(2))$	0.0534	0.9704	0.0771	0.0212	0.0007	0.0005
Actual MLT	0.2626	0.6189	1.0932	2.0225	3.0680	4.5462
$E(M\hat{L}T)$	0.3093	0.6921	1.1057	1.9969	3.1224	4.5670
$ER(M\hat{L}T)$	0.0030	0.0217	0.0074	0.0370	0.0213	0.0691
$E(MLT^*)$	0.2730	0.6608	1.0895	1.9889	3.1035	4.5489
$ER(MLT^*)$	0.0002	0.0156	0.0070	0.0303	0.0155	0.0614

From Table 1, we observe that the results of N-R method are quite satisfactory. Simulated expected values of the estimates are quite close to the corresponding actual values. Expected risks are gradually increasing with θ and the confidence intervals cover the actual values of θ quite satisfactorily. Also, we see that Bayes estimators are almost throughout much better than the corresponding MLEs in the sense of bias as well as of expected risk. This is because the additional information of prior distribution of θ is available.

6. Fitting of dMax(θ)

In lifetime experiments, the lifetime data are collected on items under study. Once the data are collected, one of the most important decisions to make is to identify or fit a suitable lifetime model. There are various methods suggested in the literature for testing the goodness of fit of a statistical distribution, such as χ^2-test, K-S test, maximum likelihood test & probability plots. Two popular methods are described below.

(i) In maximum likelihood test we suggest a distribution to be the best fit for which $L(\underset{\sim}{x}, \hat{\theta})$ is maximum. Here, $\hat{\theta}$ is the MLE of parameter θ, calculated from the sample $\underset{\sim}{x}$. Equivalently, we can minimize $\{-\log(\underset{\sim}{x}, \hat{\theta})\}$.

(ii) In χ^2-test the value of the statistic $\chi^2_{\text{calculated}} = \sum_{i=1}^{n} \frac{(O_i - E_i)^2}{E_i}$ is calculated. Here, O_i and E_i respectively denote the observed and expected frequencies of x_i. In the calculations of E_is we use the MLE $\hat{\theta}$ and pmf of the distribution to be fitted. The distribution with maximum p-value corresponding to $\chi^2_{\text{calculated}}$ is the best fit.

In view of the above methods, we now give a detailed analysis of fitting of dMW distribution and compare it with Poisson, geometric and dRayleigh distributions. In this simulation study for $\theta = 0.5$, 1.0, 2.0, 5.0, 10.0 and 20.0; random sample of size n = 15, 25, 35 and 50 were generated from dMax(θ) population using Theorem 3. Then, using the above two methods we compare the goodness of fit of these four distributions. The summary of simulation results is presented in Table 2. The values with boldface, in Table 2 show that, the sample given in column 2 fit best to the dMax(θ) distribution in almost all the cases with varying θ and sample size n.

Table 2: Table for goodness of fit

n ↓	Sample	− log L				p-value			
		dMax	Poisson	Geo.	dRay.	dMax	Poisson	Geo.	dRay.
θ = 0.5									
15	0x11, 1x4	**8.72**	9.29	9.78	8.77	**0.99**	**0.99**	0.52	**0.99**
25	0x18, 1x7	**14.9**	15.9	16.8	15.0	**0.99**	0.64	0.32	**0.99**
35	0x25, 1x10	**21.0**	22.5	23.8	21.2	**0.99**	0.70	0.42	0.70
50	0x38, 1x12	**27.6**	29.1	30.5	27.8	**0.99**	0.73	0.48	0.73
θ = 1.0									
15	0x5, 1x10	**10.6**	14.1	16.8	11.8	**0.60**	0.12	0.04	0.30
25	0x9, 1x16	**17.9**	23.1	27.4	19.5	**0.42**	0.11	0.01	0.23
35	0x11, 1x24	**24.7**	33.1	39.9	27.4	**0.17**	0.02	0.00	0.04
50	0x22, 1x26, 2x2	**41.5**	46.7	52.9	42.4	**0.99**	0.01	0.00	0.49
θ = 2.0									
15	0, 1x10, 2x4	**13.6**	17.5	22.7	15.3	**0.18**	0.01	0.00	0.06
25	0x3, 1x16, 2x6	**23.5**	29.0	36.6	25.7	**0.30**	0.01	0.00	0.03
35	0x5, 1x21, 2x8, 3	**36.3**	42.0	51.8	38.4	**0.68**	0.01	0.00	0.13
50	0x9, 1x26, 2x15	**52.5**	60.1	74.0	55.1	**0.51**	0.01	0.00	0.12
θ = 5.0									
15	1x6, 2x4, 3x2, 4x2	**21.6**	23.0	29.0	22.6	**0.53**	0.39	0.00	0.39
25	0,1x9,2x6, 3x6,4x3	**37.5**	39.1	48.1	38.5	**0.62**	0.52	0.00	0.52
35	0x3,1x9,2x11,3x8,4x4	**54.1**	55.5	67.2	57.9	**0.80**	0.46	0.00	0.73
50	0, 1x13,2x18, 3x15,4x2	**70.5**	76.1	141.6	574.8	**0.36**	0.02	0.00	0.00
θ =10.0									
15	1x3, 2x3, 3x4, 4, 5x3, 6	**26.6**	27.2	34.3	27.6	**0.96**	0.93	0.03	0.60
25	0,1x3, 2x5,3x7,4x5,5x2, 6, 7	**46.9**	47.0	56.8	47.4	**0.99**	**0.99**	0.01	0.91
35	0x3,1x2,2x7,3x10,4x6,5x5,6x2	66.8	**66.4**	159.8	67.0	0.34	**0.67**	0.00	0.21
50	0,1x3,2x11,3x18,4x11,5x3,6x2,7	**86.0**	88.2	236.9	670.3	**0.51**	0.15	0.00	0.00

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