Estimation and Optimum Constant-Stress Partially Accelerated Life Test Plans for Pareto Distribution of the Second Kind with Type-I Censoring

By

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Key Words and Phrases: Reliability; Pareto distribution; partial acceleration; constant-stress; maximum likelihood estimation; Fisher information matrix; optimum test plans; type-I censoring.

Abstract

This paper considers the case of Constant-Stress Partially Accelerated Life Testing (CSPALT) when two stress levels are involved under type-I censoring. The lifetimes of test items are assumed to follow a two-parameter Pareto lifetime distribution. Maximum Likelihood (ML) method is used to estimate the parameters of CSPALT model. Confidence intervals for the model parameters are constructed. Optimum CSPALT plans, that determine the best choice of the proportion of test units allocated to each stress, are developed. Such optimum test plans minimize the Generalized Asymptotic Variance (GAV) of the ML estimators of the model parameters. For illustration, numerical examples are presented.

1. Introduction

Under continuing quest for better, i.e. more reliable products, it is more difficult to acquire failure information quickly for products tested at the usual-use condition. In order to shorten the testing period, all or some of test units may be subjected to more severe conditions than normal ones. Such Accelerated Life Testing (ALT) or Partially Accelerated Life Testing (PALT) results in shorter lives than would be observed under normal operating conditions. In ALT, test items are run only at accelerated conditions, while in PALT they are run at both normal and accelerated conditions. As Nelson (1990) indicates, the stress can be applied in various ways, commonly used methods are step-stress and constant-stress. Under step-stress PALT, a test item is first run at use condition and, if it does not fail for a specified time, then it is run at accelerated condition until failure occurs or the observation is censored, while the constant-stress PALT runs each item at either normal use or accelerated condition only, i.e. each unit is run at a constant-stress level until the test is terminated. Accelerated test stresses involve higher than usual temperature, voltage, pressure, load, humidity, …, etc., or some combination of them. The object of a PALT is to collect more failure data in a limited time without necessarily using high stresses to all test units.

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For an overview of step-stress PALT with type-I censoring, Bai & Chung (1992) discussed both the problems of estimation and optimally designing PALT for test items having an exponential distribution. For items having lognormally distributed lives, PALT plans were developed by Bai, Chung & Chun (1993). Attia et al. (1996) considered only the estimation problem of the Weibull distribution parameters using the ML method. Concerning the constant-stress PALT, there are only three studies on the optimally designing constant-stress PALT, see Bai & Chung (1992), Bai, Chung & Chun (1993) and Ismail (2006). Abdel-Ghani (1998) considered only the estimation problem in constant-stress PALT for the Weibull distribution. In this paper, the problems of both estimation and optimal design constant-stress PALT are considered under Pareto distribution of the second kind using type-I censoring.

Nelson (1990) pointed out that the constant-stress testing has several advantages: first, it is easier to maintain a constant-stress level in most tests. Second, accelerated test models for constant-stress are better developed for some materials and products. Third, data analysis for reliability estimation is well developed. Also, as Yang (1994) indicated, constant-stress accelerated life tests are widely used to save time & money.

The paper is organized as follows: In Section 2 the used model and test method are described. The maximum-likelihood estimators of the model parameters are obtained in Section 3. Section 4 is devoted to the derivation of the confidence intervals for the model parameters. In Section 5 optimal plans for simple constant-stress PALT are developed. Numerical examples are given in Section 6 to illustrate the theoretical results.

**2. The Model and Test Method**

**Notation:**

- \( n \) total number of test items in a PALT
- \( \eta \) censoring time of a PALT
- \( T \) lifetime of an item at use condition
- \( X \) lifetime of an item at accelerated condition
- \( \beta \) acceleration factor (\( \beta > 1 \))
- \( P_u \) probability that an item tested only at use condition fails by \( \eta \)
- \( P_a \) probability that an item tested only at accelerated condition fails by \( \eta \)
- \( \wedge \) implies a maximum likelihood estimate
- \( \theta \) Pareto scale parameter
- \( \alpha \) Pareto shape parameter
- \( t_i \) observed lifetime of item \( i \) tested at use condition
- \( x_j \) observed lifetime of item \( j \) tested at accelerated condition
- \( \delta_{ui}, \delta_{uj} \) indicator functions: \( \delta_{ui} \equiv I(T_i \leq \eta), \delta_{uj} \equiv I(X_j \leq \eta) \)
- \( \pi \) proportion of sample units allocated to accelerated condition
- \( \pi^* \) optimum proportion of sample units allocated to accelerated condition
numbers of items failed at use and accelerated conditions, respectively

\[ t^{(i)} \leq \ldots \leq t^{(n_a)} \leq \eta \]

ordered failure times at use condition

\[ x^{(i)} \leq \ldots \leq x^{(n_a)} \leq \eta \]

ordered failure times at accelerated condition

### 2.1 The Pareto Distribution: As a Lifetime model

The lifetimes of the test items are assumed to follow a two-parameter Pareto distribution of the second kind. The Pareto distribution was introduced by Pareto (1987) as a model for the distribution of income. In recent years, its models in several different forms have been studied by many authors Davis and Feldstein (1979), Cohen and Whitten (1988), Grimshaw (1993) among others. The Pareto distribution of the second kind also known as Lomax or Pearson's Type VI distribution (Johnson et al., 1994) has been found to provide a good model in biomedical problems, such as survival time following a heart transplant (Bain and Engelhardt, 1992). Using the Pareto distribution, Dyer (1981) studied annual wage data of production line workers in a large industrial firm. Lomax (1954) used this distribution in the analysis of business failure data. The length of wire between flaws also follows a Pareto distribution (Bain and Engelhardt, 1992). Since the Pareto distribution has a decreasing hazard or failure rate, it has often been used to model incomes and survival times (Howlader and Hossain, 2002).

The probability density function of the Pareto distribution of the second kind is given by

\[
f_T(t; \theta, \alpha) = \frac{\alpha \theta^\alpha}{(\theta + t)^{\alpha+1}} \quad ; \ t > 0, \ \theta > 0, \ \alpha > 0, \quad (1)
\]

The survival function takes the form:

\[
R(t) = \frac{\theta^\alpha}{(\theta + t)^\alpha}, \quad (2)
\]

and the corresponding hazard or instantaneous failure rate is as follows:

\[
h(t) = \frac{\alpha}{\theta + t} \quad (3)
\]

As indicated by McCune and McCune (2000), the Pareto distribution has classically been used in economic studies of income, size of cities and firms, service time in queuing systems and so on. Also, it has been used in connection with reliability theory and survival analysis (Davis & Feldstein, 1979).

### 2.2 Constant-Stress PALT

The test procedure of the constant-stress PALT and its assumptions are described as follows:

- **Test Procedure**

In a constant-stress PALT, the total sample size \( n \) of test units is divided into two parts such that:

1. \( n\pi \) items randomly chosen among \( n \) test items sampled are allocated to accelerated condition and the remaining are allocated to use condition.
2. Each test item is run until censoring time $\eta$ and the test condition is not changed.

- **Assumptions**
  1. The lifetimes $T_i, i = 1, \ldots, n(1-\pi)$ of items allocated to use condition, are i.i.d. r.v.'s.
  2. The lifetimes $X_j, j = 1, \ldots, n\pi$ of items allocated to accelerated condition, are i.i.d r.v.'s.
  3. The lifetimes $T_i$ and $X_j$ are mutually statistically-independent.

**3. Computing Maximum Likelihood Parameter-Estimates**

Maximum likelihood estimators (MLEs) of the parameters are used, since they are asymptotically normally distributed and asymptotically efficient in many cases (Grimshaw, 1993). Also, Bugaighis (1988) indicated that the ML procedure generally yields efficient estimators. However, these estimators do not always exist in closed form, so numerical techniques are used to compute them.

In a simple constant-stress PALT, the test item is run either at use condition or at accelerated condition only. A simple constant-stress test uses only two stresses and allocates the $n$ sample units to them (Miller & Nelson, 1983).

Since the lifetimes of the test items follow Pareto distribution of the second kind, the probability density function of an item tested at use condition is given by:

$$f_T(t) = \frac{\alpha \theta^\alpha}{(\theta + t)^{\alpha+1}}, \ t \geq 0$$

(4)

While for an item tested at accelerated condition, the probability density function is given by:

$$f_X(x) = \frac{\beta \alpha \theta^\alpha}{(\theta + \beta x)^{\alpha+1}}, \ x \geq 0$$

(5)

where $X = \beta^{-1} T$

The likelihood for $(t_i, \delta_{ui})$, the likelihood for $(x_j, \delta_{aj})$ and the total likelihood for $(t_i; \delta_{ui}, \ldots, t_{n(1-\pi)}; \delta_{un(1-\pi)}, x_i; \delta_{ui}, \ldots, x_{n\pi}; \delta_{an\pi})$ are respectively as follows:

$$L_{uii}(t_i, \delta_{ui} / \theta, \alpha) = \left[\frac{\alpha \theta^\alpha}{(\theta + t_i)^{\alpha+1}}\right]^{\delta_{ui}} \left[\frac{\theta^\alpha}{(\theta + \eta)}\right]^{\delta_{ui}}$$

(6)
\[ L_{aj}(x_j, \delta_{aj} / \beta, \theta, \alpha) = \left[ \frac{\beta \alpha \theta^\alpha}{(\theta + \beta x_j)^{\alpha+1}} \right]^{\delta_{aj}} \left[ \frac{\theta^\alpha}{(\theta + \beta \eta)^\alpha} \right]^{\delta_{aj}}, \]  

(7)

\[ L(\xi, x / \beta, \theta, \alpha) = \prod_{i=1}^{n(1-\pi)} L_{ui}(t_i, \delta_{ui} / \theta, \alpha) \prod_{i=1}^{n} L_{aj}(x_j, \delta_{aj} / \beta, \theta, \alpha) \]

\[ = \prod_{i=1}^{n(1-\pi)} \left[ \frac{\alpha \theta^\alpha}{(\theta + t_i)^{\alpha+1}} \right]^{\delta_{ui}} \left[ \frac{\theta^\alpha}{(\theta + \eta)^\alpha} \right]^{\delta_{ui}} \]

\[ = \prod_{i=1}^{n(1-\pi)} \left[ \frac{\beta \alpha \theta^\alpha}{(\theta + \beta x_j)^{\alpha+1}} \right]^{\delta_{aj}} \left[ \frac{\theta^\alpha}{(\theta + \beta \eta)^\alpha} \right]^{\delta_{aj}} \]

(8)

Where \( \delta_{ui} = 1 - \delta_{ui} \) and \( \delta_{aj} = 1 - \delta_{aj} \).

It is usually easier to maximize the natural logarithm of the likelihood function rather than the likelihood function itself. The first derivatives of the natural logarithm of the total likelihood function in (8) with respect to \( \beta, \theta \) and \( \alpha \) are given by:

\[ \frac{\partial \ln L}{\partial \beta} = \frac{n_a - (n\pi - n_a)\alpha \eta}{\beta} - (\alpha + 1) \sum_{j=1}^{n} \delta_{aj} \frac{x_j}{\theta + \beta x_j} \]

(9)

\[ \frac{\partial \ln L}{\partial \theta} = \frac{n\alpha - (n\pi - n_a)\alpha}{\theta} - (n\pi - n_u)\alpha \frac{\alpha}{\theta + \eta} - (\alpha + 1) \left[ \sum_{i=1}^{n(1-\pi)} \frac{\delta_{ui}}{\theta + t_i} + \sum_{j=1}^{n} \frac{\delta_{aj}}{\theta + \beta x_j} \right] \]

(10)
\[
\frac{\partial \ln L}{\partial \alpha} = \frac{n_u + n_a}{\alpha} + n \ln \theta - (n \pi - n_a) \ln(\theta + \beta \eta) - (n(1 - \pi) - n_u) \ln(\theta + \eta)
\]
\[
- \left[ \sum_{i=1}^{n(1-\pi)} \delta_{ui} \ln(\theta + t_i) + \sum_{j=1}^{n \pi} \delta_{aj} \ln(\theta + \beta \kappa_j) \right]
\]
\[
\frac{\partial \ln L}{\partial \beta} = \frac{n_u + n_a}{S} - \left( \frac{n_u + n_a}{S} + 1 \right) \sum_{j=1}^{n \pi} \delta_{aj} \frac{x_j}{\hat{\theta} + \hat{\beta} \kappa_j} = 0
\]
\[
\frac{\partial \ln L}{\partial \theta} = \frac{n(n_u + n_a)}{S} - \left( \frac{n(n_u + n_a)}{S} + \frac{n \pi - n_a}{\hat{\theta} + \hat{\beta} \eta} \right) - \left( \frac{n(1 - \pi) - n_u}{\hat{\theta} + \eta} \right) \frac{n_u + n_a}{S}
\]
\[
- \left( \frac{n_u + n_a}{S} + 1 \right) \left[ \sum_{i=1}^{n(1-\pi)} \delta_{ui} \frac{1}{\hat{\theta} + t_i} + \sum_{j=1}^{n \pi} \delta_{aj} \frac{1}{\hat{\theta} + \hat{\beta} \kappa_j} \right] = 0
\]

The ML estimates of the parameters are the values of $\beta$, $\theta$ and $\alpha$ which solve the equations obtained by letting each of them be zero. So, from the last equation, the ML estimate of $\alpha$ is given by:

\[
\hat{\alpha} = \frac{n_u + n_a}{S}
\]

where

\[
S = \frac{n(1-\pi)}{\delta_{ui}} \ln(\hat{\theta} + t_i) + \frac{n \pi}{\delta_{aj}} \ln(\hat{\theta} + \hat{\beta} \kappa_j) + (n \pi - n_a) \ln(\hat{\theta} + \beta \eta)
\]

\[+ (n(1 - \pi) - n_u) \ln(\hat{\theta} + \eta) - n \ln \hat{\theta}
\]

By substituting for $\alpha$ into the two equations (9) and (10) and equating each of them to zero, the system equations are reduced into the following two non-linear equations:
Obviously, it is very difficult to obtain a closed-form solution for the two non-linear equations (13) and (14). So, iterative procedures must be used to solve these equations, numerically. The Newton-Raphson method is used to determine the ML estimates of $\beta$ and $\theta$. Thus, once the values of $\beta$ and $\theta$ are determined, an estimate of $\alpha$ is easily obtained from (12).

In relation to the asymptotic variance-covariance matrix of the ML estimators of the parameters, it can be approximated by numerically inverting the Fisher-information matrix $F$. It is composed of the negative second derivatives of the natural logarithm of the likelihood function evaluated at the ML estimates. Therefore, the asymptotic Fisher-information matrix can be written as follows:

$$ F = \begin{bmatrix} -\frac{\partial^2 \ln L}{\partial \beta^2} & -\frac{\partial^2 \ln L}{\partial \beta \partial \theta} & -\frac{\partial^2 \ln L}{\partial \beta \partial \alpha} \\ -\frac{\partial^2 \ln L}{\partial \theta \partial \beta} & -\frac{\partial^2 \ln L}{\partial \theta^2} & -\frac{\partial^2 \ln L}{\partial \theta \partial \alpha} \\ -\frac{\partial^2 \ln L}{\partial \alpha \partial \beta} & -\frac{\partial^2 \ln L}{\partial \alpha \partial \theta} & -\frac{\partial^2 \ln L}{\partial \alpha^2} \end{bmatrix} \downarrow (\hat{\beta}, \hat{\theta}, \hat{\alpha}) $$

(15)

The elements of the above matrix $F$ can be expressed by the following equations:

$$ \frac{\partial^2 \ln L}{\partial \beta^2} = -\frac{n_a}{\beta^2} + \frac{(n\pi - n_a)\alpha \eta}{(\theta + \beta \eta)^2} + (\alpha + 1) \sum_{j=1}^{n\pi} \delta_{aj} \frac{x_j}{(\theta + \beta \xi_j)^2} $$

(16)

$$ \frac{\partial^2 \ln L}{\partial \beta \partial \theta} = \frac{(n\pi - n_a)\alpha \eta}{(\theta + \beta \eta)^2} + (\alpha + 1) \sum_{j=1}^{n\pi} \delta_{aj} \frac{x_j}{(\theta + \beta \xi_j)^2} $$

(17)

$$ \frac{\partial^2 \ln L}{\partial \beta \partial \alpha} = -\frac{(n\pi - n_a)\eta}{\theta + \beta \eta} - \frac{n\pi}{\delta_{aj}} \frac{x_j}{\theta + \beta \xi_j} $$

(18)

$$ \frac{\partial^2 \ln L}{\partial \theta^2} = \frac{n\alpha}{\theta^2} + \frac{(n\pi - n_a)\alpha}{(\theta + \beta \eta)^2} + \frac{(n(1 - \pi) - n_u)\alpha}{(\theta + \eta)^2} + $$

$$ (\alpha + 1) \left[ \sum_{i=1}^{n(1-\pi)} \frac{\delta_{ui}}{(\theta + \xi_i)^2} + \sum_{j=1}^{n\pi} \frac{\delta_{aj}}{(\theta + \beta \xi_j)^2} \right] $$

(19)
\[
\frac{\partial^2 \ln L}{\partial \theta \partial \alpha} = - \frac{n}{\theta + \beta \eta} \left( n \pi - n_u \right) - \frac{n(1 - \pi) - n_u}{\theta + \eta} \\
- \left[ \sum_{i=1}^{n(1-\pi)} \frac{\delta_{ui}}{\theta + \xi_i} + \sum_{j=1}^{n \pi} \frac{\delta_{aj}}{\theta + \beta \xi_j} \right]
\]

\[
\frac{\partial^2 \ln L}{\partial \alpha^2} = - \frac{n_u + n_a}{\alpha^2}
\]

Consequently, the maximum likelihood estimators of \( \beta, \theta \) and \( \alpha \) have an asymptotic variance-covariance matrix defined by inverting the Fisher information matrix \( F \) as indicated before.

4. Confidence Intervals for the Model Parameters

As indicated by Vander Wiel and Meeker (1990), the most common method to set confidence bounds for the parameters is to use the large-sample (asymptotic) normal distribution of the ML estimators.

To construct a confidence interval for a population parameter \( \lambda \); assume that \( L_{\lambda} = L_{\hat{\lambda}}(y_1, ..., y_n) \) and \( U_{\lambda} = U_{\hat{\lambda}}(y_1, ..., y_n) \) are functions of the sample data \( y_1, ..., y_n \) such that

\[
P_{\lambda}(L_{\lambda} \leq \lambda \leq U_{\lambda}) = \gamma,
\]

where the interval \([L_{\lambda}, U_{\lambda}]\) is called a two-sided \( \gamma \) 100 % confidence interval for \( \lambda \). \( L_{\lambda} \) and \( U_{\lambda} \) are the lower and upper confidence limits for \( \lambda \), respectively. The random limits \( L_{\lambda} \) and \( U_{\lambda} \) enclose \( \lambda \) with probability \( \gamma \).

Asymptotically, the maximum likelihood estimators, under appropriate regularity conditions, are consistent and normally distributed. Therefore, the two-sided approximate \( \gamma \) 100 % confidence limits for a population parameter \( \lambda \) can be constructed such that:

\[
P\left[- z \leq \frac{\hat{\lambda} - \lambda}{\sigma(\hat{\lambda})} \leq z \right] = \gamma,
\]

where \( z \) is the \([100(1-\gamma/2)]^{th}\) standard normal percentile. Therefore, the two-sided approximate \( \gamma \) 100 % confidence limits for \( \beta, \theta \) and \( \alpha \) are given respectively as follows:
\[ L_\beta = \hat{\beta} - z\sigma(\hat{\beta}) \quad U_\beta = \hat{\beta} + z\sigma(\hat{\beta}) \]
\[ L_\theta = \hat{\theta} - z\sigma(\hat{\theta}) \quad U_\theta = \hat{\theta} + z\sigma(\hat{\theta}) \]
\[ L_\alpha = \hat{\alpha} - z\sigma(\hat{\alpha}) \quad U_\alpha = \hat{\alpha} + z\sigma(\hat{\alpha}) \]

5. Optimum Simple Constant-Stress Test Plans

Most of the test plans are equally-spaced test stresses, i.e. the same number of test units are allocated to each stress. Such test plans are usually inefficient for estimating the mean life at design stress (Yang, 1994). In this section, test plans statistically optimum are developed to decide the optimal sample-proportion allocated to each stress. Therefore, to determine the optimal sample-proportion \( \pi^* \) allocated to accelerated condition, \( \pi \) is chosen such that the Generalized Asymptotic Variance (GAV) of the ML estimators of the model parameters is minimized. The GAV of the ML estimators of the model parameters as an optimality criterion is commonly used and defined below as the reciprocal of the determinant of the Fisher-information matrix \( F \) (Bai, Kim & Chun, 1993). That is,

\[ \text{GAV}(\hat{\beta}, \hat{\theta}, \hat{\alpha}) = \frac{1}{|F|} \]

The Newton-Raphson method is applied to numerically determine the best choice of the sample-proportion allocated to accelerated condition which minimizes the GAV as defined before. Accordingly, the corresponding expected optimal numbers of items failed at use and accelerated conditions can be obtained, respectively, as follows:

\[ n^* = n(1 - \pi^*) P \quad \text{and} \quad n^* = n\pi^* P \]

where

\[ P_{\text{u}} = \frac{1 - \frac{\hat{\alpha}}{(\hat{\theta} + \eta)}}{(\hat{\theta} + \hat{\eta})} \quad \text{and} \quad P_{\text{a}} = \frac{1 - \frac{\hat{\alpha}}{(\hat{\theta} + \hat{\eta})}}{(\hat{\theta} + \hat{\eta})} \]

6. Simulation Studies

The main objective of this section is to make a numerical investigation for illustrating the theoretical results of both estimation and optimal design problems. Several data sets are generated from Pareto distribution for different combinations of the true parameter values of \( \beta, \theta \) and \( \alpha \) and for sample sizes 100, 200, 300, 400 and 500 using 500 replications for each sample size. The true parameter values used in this simulation study are \((2, 5, 0.5)\) and \((4, 7, 0.7)\). Computer programs are prepared and the Newton-Raphson method is used for the practical application of the ML estimators of \( \beta, \theta \) and \( \alpha \). Therefore, the derived nonlinear logarithmic likelihood equations in (13) and (14) are solved iteratively. Once the values of \( \beta \) and \( \theta \) are determined, an estimate of the shape parameter \( \alpha \) is easily obtained from equation (12). For different sample sizes and true values of the parameters,
the ML estimates, estimated asymptotic variances and confidence intervals for parameter-estimates are reported in Tables (1) and (3). While in Tables (2) and (4), the optimal sample-proportion $\pi^*$ allocated to accelerated condition, the expected optimal numbers of items failed at use and accelerated conditions and the optimal GAV of the ML estimators of the model parameters are included.

Results of simulation studies provide insight into the sampling behavior of the estimators. The numerical results indicate that the ML estimates approximate the true values of the parameters as the sample size $n$ increases. Also, as shown from the numerical results, the asymptotic variances of the estimators decrease as the sample size $n$ is getting to be larger. The equations in (24) are used to construct the approximate confidence limits for the three parameters $\beta, \theta$ and $\alpha$, with results shown in Tables (1) and (3). These Tables present 2-sided approximate confidence bounds based on 95% confidence degree for the parameters. As seen from the results, the intervals of the parameters appear to be narrow as the sample size $n$ increases.

Also, optimum test plans are developed numerically. It can be observed from the numerical results, via $\pi^*$, presented in Tables (2) and (4), that the optimum test plans do not allocate the same number of test units to each stress. In practice, the optimum test plans are important for improving precision in parameter estimation and thus improving the quality of the inference. So, these optimum plans are more useful and more efficient for estimating the life distribution at design stress. Also, Tables (2) and (4) include the expected numbers of items failed at use and accelerated conditions represented by $n_u^*$ and $n_a^*$, respectively, for each sample size. Finally, these Tables also present the optimal GAV of the ML estimators of the model parameters which is obtained numerically with $\pi^*$ in place of $\pi$ for different sized samples. As indicated from the results, the optimal GAV decreases as the sample size $n$ increases.

### Table (1)

The ML estimates, estimated asymptotic variances of the ML estimators and confidence bounds of the parameters ($\beta, \theta, \alpha$) set at (2, 5, 0.5), respectively, given $\pi = 0.25$ and $\eta = 50$ for different sized samples under type-I censoring in constant-stress PALT.

<table>
<thead>
<tr>
<th>$n$</th>
<th>Parameter</th>
<th>Estimate</th>
<th>Variance</th>
<th>Lower bound</th>
<th>Upper bound</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>$\beta$</td>
<td>2.3389</td>
<td>2.1740</td>
<td>0.5510</td>
<td>5.2288</td>
</tr>
<tr>
<td></td>
<td>$\theta$</td>
<td>3.7989</td>
<td>3.3554</td>
<td>0.2194</td>
<td>7.3785</td>
</tr>
<tr>
<td></td>
<td>$\alpha$</td>
<td>0.4832</td>
<td>0.0127</td>
<td>0.2621</td>
<td>0.7044</td>
</tr>
<tr>
<td>200</td>
<td>$\beta$</td>
<td>2.2703</td>
<td>0.8389</td>
<td>0.4751</td>
<td>4.0654</td>
</tr>
<tr>
<td></td>
<td>$\theta$</td>
<td>4.4933</td>
<td>2.2644</td>
<td>1.5439</td>
<td>7.4427</td>
</tr>
<tr>
<td></td>
<td>$\alpha$</td>
<td>0.4953</td>
<td>0.0067</td>
<td>0.3347</td>
<td>0.6559</td>
</tr>
<tr>
<td>300</td>
<td>$\beta$</td>
<td>2.2002</td>
<td>0.4755</td>
<td>0.8486</td>
<td>3.5519</td>
</tr>
<tr>
<td></td>
<td>$\theta$</td>
<td>4.7169</td>
<td>1.6315</td>
<td>2.2134</td>
<td>7.2204</td>
</tr>
<tr>
<td></td>
<td>$\alpha$</td>
<td>0.5004</td>
<td>0.0046</td>
<td>0.3681</td>
<td>0.6327</td>
</tr>
<tr>
<td>400</td>
<td>$\beta$</td>
<td>2.1688</td>
<td>0.3424</td>
<td>1.0219</td>
<td>3.3156</td>
</tr>
<tr>
<td></td>
<td>$\theta$</td>
<td>4.8499</td>
<td>1.2957</td>
<td>2.6189</td>
<td>7.0809</td>
</tr>
<tr>
<td></td>
<td>$\alpha$</td>
<td>0.5018</td>
<td>0.0035</td>
<td>0.3864</td>
<td>0.6173</td>
</tr>
<tr>
<td>500</td>
<td>$\beta$</td>
<td>2.1412</td>
<td>0.2617</td>
<td>1.1386</td>
<td>3.1439</td>
</tr>
<tr>
<td></td>
<td>$\theta$</td>
<td>4.9363</td>
<td>1.0640</td>
<td>2.9145</td>
<td>6.9580</td>
</tr>
<tr>
<td></td>
<td>$\alpha$</td>
<td>0.5019</td>
<td>0.0028</td>
<td>0.3987</td>
<td>0.6051</td>
</tr>
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</table>
Table (2)
The results of optimal design of the life test for different sized samples under type-I censoring in constant-stress PALT

<table>
<thead>
<tr>
<th>n</th>
<th>$\pi^*$</th>
<th>$n_{u}$</th>
<th>$n_{a}$</th>
<th>Optimal GAV</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>0.4338</td>
<td>41</td>
<td>35</td>
<td>3.6492</td>
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<td>285</td>
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Table (3)
The ML estimates, estimated asymptotic variances of the ML estimators and confidence bounds of the parameters ($\beta$, $\theta$, $\alpha$) set at (4, 7, 0.7), respectively, given $\pi = 0.25$ and $\eta = 50$ for different sized samples under type-I censoring in constant-stress PALT

<table>
<thead>
<tr>
<th>n</th>
<th>Parameter</th>
<th>Estimate</th>
<th>Variance</th>
<th>Lower bound</th>
<th>Upper bound</th>
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<tr>
<td>100</td>
<td>$\beta$</td>
<td>4.7223</td>
<td>5.2396</td>
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<td>9.2088</td>
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<tr>
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<td>$\theta$</td>
<td>6.9668</td>
<td>9.2383</td>
<td>1.0094</td>
<td>12.9241</td>
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<tr>
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<td>0.0359</td>
<td>0.3997</td>
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<td>1.4753</td>
<td>7.2807</td>
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<td>11.2314</td>
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<td>0.4812</td>
<td>0.9685</td>
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<td>0.0104</td>
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<td>0.0058</td>
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Table (4)
The results of optimal design of the life test for different sized samples under type-I censoring in constant-stress PALT

<table>
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<tr>
<th>n</th>
<th>$\pi^*$</th>
<th>$n_{u}$</th>
<th>$n_{a}$</th>
<th>Optimal GAV</th>
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</thead>
<tbody>
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<tr>
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<td>0.7619</td>
<td>92</td>
<td>347</td>
<td>0.0069</td>
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</table>
References


