The NBUTA Class of Life Distributions
by

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Abstract : A new class of life distributions, namely new better than used in the total time on test on average transform ordering (TTTA)**, is introduced. The relationship of this class to other classes of life distributions, and closure properties under some reliability operations, are discussed. We provide a simple argument based on stochastic orders that the class is closed under the formation of series systems in case of independent identically distributed components. Behavior of this class is developed in terms of the monotonicity of the residual life of k -out-of n systems given the time at which the** (n-k)**-th failure has occurred.**
Finally, we discuss testing exponentially against the aging property.

Key words—increasing concave order, k out-of- n systems, life testing, mixing, random minima, series system, stochastic order, transform order.

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I. INTRODUCTION AND MOTIVATION

Because the accurate distribution of the life of an element or a system is often unavailable practically, nonparametric aging properties are quite useful for modeling aging or wear-out processes, and for constructing maintenance policies. Such aging classes are derived via several notions of comparison between random variables. The most commonly used comparison include Muller and Stoyan (2002), Shaked and Shanthikumar (1994), the stochastic comparison, and the increasing concave comparison. Formally, if *X* & *Y* are two random variables with distributions *F* & *G* (survivals $\bar{F}$ & $\bar{G}$), respectively, then we say that *X* is smaller than *Y* in the

(i) Stochastic order sense (denoted by $X \le_{st} Y$), if

$$E[\phi(X)] \le E[\phi(Y)]$$

for all increasing functions ϕ; and

(ii) Increasing concave order sense (denoted by $X \le_{icv} Y$), if

$$E[\phi(X)] \le E[\phi(Y)]$$

for all increasing concave functions ϕ.

Consider a distribution function *F*, of a nonnegative random variable *X*, which is strictly increasing on its interval support.
Let $p \in (0,1)$, and $t \ge 0$ be two values related by $t = F^{-1}(p)$,where F^{-1} is the right continuous inverse of *F*. Denote by

$$A_F \equiv \{(x,u): u \in (0,p), x \in (0, F^{-1}(u))\}$$

and

$$B_F \equiv \{(x,u): u \in (p,1), x \in (0, F^{-1}(u))\}$$

The above areas of the regions have various intuitive meanings in different applications. For example, if F is the distribution function of the lifetime of a machine, then

$$T_X(p) \equiv \| A_F(p) \cup B_F(p) \|, p \in (0,1)$$

Corresponds to the total time on test (*TTT*) transform associated with this distribution. The TTT-order is usually denoted by $X \underset{ttt}{\le} Y$ and

$$X \underset{ttt}{\le} Y \Leftrightarrow \int_0^{F^{-1}(p)} \bar{F}(x)dx \le \int_0^{G^{-1}(p)} \bar{G}(x)dx, p \in (0,1)$$

Recently, *the TTT*-transform order has also come to use in reliability and life testing Kochar et al. (2002), and Ahmad et al. (2006). Applications, properties, and interpretations of the *TTT*-transform order in the statistical theory of reliability, and in economics can be found in Kochar et al.(2002), Li and Zuo (2002), Ahmad and Kayid (2005), and Ahmad et al. (2001).
In this paper we introduce and study a closely related order; namely the total time on test on average (TTTA) order.

We say that a random variable X is smaller than a random variable Y in the TTTA-transform order (denoted by $X \leq_{ttta} Y$) if, and only if

$$\mathrm{T}^*{}_{\mathrm{X}} = \int_0^1 T_X(p)dp = \int_0^1 \int_0^{F^{-1}(p)} \bar{F}(x)dxdp \leq \int_0^1 \int_0^{G^{-1}(p)} \bar{G}(x)dxdp = \int_0^1 T_Y(p)dp = \mathrm{T}^*{}_{\mathrm{Y}}$$

Where, F^{-1}, G^{-1} are the right continuous inverses of F , and G.

Some properties of the TTTA-order.

Property 1: The order $\underset{ttta}{\leq}$ is not location independent, i.e.,

$X \underset{ttta}{\leq} Y \not\Rightarrow X \underset{ttta}{\leq} Y + c$ for any $c \in (-\infty, \infty)$.

However, if Y is a random variable with distribution function G, then

$T^*_{Y+c} = T^*_Y + c$. To see it, we note that:

$T_{Y+c}(p) = \left\| A_{G(.-c)}(p) \cup B_{G(.-c)}(p) \right\| = \left\| A_G(p) \cup B_G(p) \right\| + c$

$= T_Y(p) + c, p \in (0,1), c \in (-\infty, \infty)$

It follows that

$\int_0^1 T_{Y+c}(p)dp = \int_0^1 T_Y(p) + c$; i.e.,

$T^*_{Y+c} = T^*_Y + c$

This means that the order $\underset{ttta}{\leq}$ is closed under right shifts of the larger variable; i.e.,

$X \underset{ttta}{\leq} Y \Rightarrow X \underset{ttta}{\leq} Y + c$ for any $c > 0$.

Property 2: $X \underset{st}{\leq} Y \Rightarrow X \underset{ttta}{\leq} Y$, i.e., The *TTTA*-transform order is weaker than the stochastic order ($\leq_{st}$)

Proof: obvious and hence is omitted.

In the context of lifetime distributions, some of the above orderings of distributions have been used to give characterizations and new definitions of aging classes. By aging, we mean the phenomenon whereby an older system has a shorter remaining lifetime, in some statistical sense, than a younger one (Bryson& Siddiqui (1969)). One of the most important approaches to the study of aging is based on the concept of the residual life. For any random variable X , let

$X_t = [X - t \mid X > t], t \in \{x : F(x) < 1\}$

denote a random variable whose distribution is the same as the conditional distribution of $X-t$ given that $X>t$. When X is the lifetime of a device, X_t can be regarded as the residual lifetime of the device at time t, given that the device has survived up to time t. We say that X is

(i) New better than used (denoted by $X \in NBU$) if $X_t \leq_{st} X$ for all $t \geq 0$ or

(ii) New better than used in the increasing concave order (denoted by $X \in NBUA$) if $X_t \leq_{icva} X$ for all $t \geq 0$.

The classes NBU, NBU(2), and NBUCA have proven to be very useful in performing analyses of life lengths, as well as usable in replacement policies. Hence, a lot of results related to these two classes have been obtained in the literature (Bryson & Siddiqui (1969), Barlow and Proschan (1981), Deshpande et al. 1986), Li and Kochar (2001), Franco et al. (2001), Loh (1984) and Hu and Xie (2002).

Recently, Ahmad et al. (2006), have used the TTT-order to define a new family of life distributions; namely the NBUT family. They have also demonstrated its usefulness in applications. It seems, however, that the NBUT property is too strong to verify as an aging property, since it has to hold for every p in the interval (0,1). Hence, one is motivated to find an aging class in between these two classes, and that is precisely what we do in the current work with the NBUTA class.

Suppose F is the distribution function of the lifetime of a unit, which could be a living organism or a mechanical component or a system. To determine whether the component is aging on average with time for the *TTTA*-transform associated with this distribution, we need to compare the average lifetime of a component $\int_0^1 T_X(p)\,dp$ with its average residual life $\int_0^1 T_{X_t}(p)\,dp$ at different ages, and hence another class corresponding to the *TTT*A-transform order is needed. Motivated by this, we propose a new aging notion as follow:

Definition 1.1*:* A random variable X, or its distribution F, is said to be new better than used in the total time on test transform order denoted by *NBUTA* if $X_t \leq_{ttta} X$ Equivalently, $X \in NBUTA$ if, and only if

$$\int_0^1 \int_0^{F_t^{-1}(p)} \bar{F}(u+t)\,du\,dp \leq \bar{F}(t) \int_0^1 \int_0^{F_t^{-1}(p)} \bar{F}(u)\,du\,dp \text{ for all } t \geq 0 \qquad (1.2)$$

The relations between the above aging classes are the following:

$$NBU \Rightarrow NBUT \Rightarrow NBUTA$$

In general, the converse of the above relation is not necessarily true. *Example* 3.2 in *Kochar et al.(*2002) shows that

$$X \leq_{ttt} Y \not\Rightarrow X \leq_{st} Y$$

and hence we conclude that the $NBUTA \not\Rightarrow NBUT$. Again, from the fact that the *TTT*-transform order is a stochastic order that combines the comparison of location with the comparison of variation, together with *Theorem* 2.2. in *Kochar et al. (2002)*, we also conclude that $NBU(2) \not\Rightarrow NBUTA$. Hence, our class is a nontrivial & practical aging class which is wider than the NBUT class.

Next, in Section II, several other properties of the class NBUTA are presented, including the preservation under some reliability operations. In Section III, behavior of the NBUTA aging notion is developed in terms of the monotonicity of the residual life of *k*-out-of-*n* systems, given the time of the (*n*-*k*)-th failure. Also, in that section, a similar conclusion based on the residual life of parallel systems is presented as well. Finally, in Section IV, we address the question of testing whether Ho is: F is exponential against H_1:
F ε NBUTA
and not exponential.

II. PRESERVATION PROPERTIES

Useful properties of aging classes of life distributions are the closure with respect to typical reliability operations (see, e.g., *Barlow*, and *Proschan*, (1981)). In this section we present some preservation results for the *NBUT*A class under random minima, series systems, and mixture.
Let X_1, X_2, … be a sequence of independent & identically distributed (*i.i.d*)random variables, and *N* be a positive integer valued random variable, which is independent of the X_i. Put

$$X_{1:N} \equiv \min\{X_1, X_2, ..., X_N\}$$

$$X_{N:N} \equiv \max\{X_1, X_2, ..., X_N\}$$

The random variables $X_{(1:N)}$,and $X_{(N:N)}$ arise naturally in reliability theory as the lifetimes of series, and parallel systems, respectively, with the random number *N* of identical components with lifetimes X_1, X_2, …, X_N. In life-testing, if a random censoring is adopted, then the completely observed data constitute a sample of random size X_1, X_2, … X_N, say, where $N > 0$ is a random variable of integer value. In actuarial science, the claims received by an insurer in a certain time interval should also be a sample of random size, and $X_{(N:N)}$ denotes the largest claim amount of the period. Also $X_{(1:N)}$ arises naturally in survival analysis as the minimal survival time of a transplant operation, where *N* of them are defective, and hence may cause death.

Some authors have made efforts to investigate preservation properties of some stochastic orders under random minima and maxima, while others have centered their attention on investigating the behavior of aging properties in coherent structure, parallel (series) systems, convolution, mixture, and renewal process (Bartoszewicz (2001), Marshall and Proschan (1972), Li and Zuo (2004), Ahmad and Kayid (2005).

In the following, we give preservation results for the class under formation of random minima. First, we recall the following modified version of a result about the preservation of the TTTA -transform order (see Li and Zuo (2004)).

Theorem 2.1: Let X_1 , X_2 , … , and Y_1 , Y_2 , … each be a sequence of i.i.d. non-negative random variables having common left end point 0; and N be independent of the X_i, and Y_i. If $X_i \leq_{ttta} Y_i$ for i = 1,2,…, Then

$$\min\{X_1, X_2, ..., X_N\} \leq_{ttta} \min\{Y_1, Y_2, ..., Y_N\}$$

Next, we give the result.

Theorem 2.2 : Let X_1 , X_2 , …be a sequence of i.i.d. lives, and N be independent of the X_i .

If X_1 is NBUTA, then min{X_1 , X_2 , …, X_N } is also NBUTA property.

Proof:

If X_1 is NBUTA, then for all $t \geq 0$

$$(X_i)_t \leq_{ttta} X_i$$

By Theorem 2.1, we have for all $t \geq 0$

$$\min\{(X_1)_t, ..., (X_N)_t\} \leq_{ttta} \min\{X_1, ..., X_N\}$$

Because for any integer n

$$\min\{(X_1)_t, ..., (X_n)_t\} \stackrel{st}{=} \left(\min\{X_1, ..., X_n\}\right)_t, t \geq 0$$

It also holds that

$$\min\{(X_1)_t, ..., (X_N)_t\} \stackrel{st}{=} \left(\min\{X_1, ..., X_N\}\right)_t, t \geq 0$$

and hence

$$\left(\min\{X_1, ..., X_N\}\right)_t \leq_{ttta} \min\{Y_1, ..., Y_N\}, t \geq 0 .$$

That is to say, $\min\{X_1, ..., X_N\}$ is NBUTA too.

On the other hand, it is a well known fact that some aging notions are preserved under the formation of a parallel or series system (see Barlow & Proschan (1981), Abouammoh and El-Neweihi (1986), Hendi et al. (1993), Li and Kochar (2001), and Pellerey and Petakos (2002). According to Theorem 2.2, Corollary 2.1 can be deduced as below.

Corollary 2.1*:* Let $X_1, X_2, \ldots, X_n$ be a set of NBUTA independent identically distributed components, and consider $T_n = \min\{X_1, X_2, \ldots, X_n\}$ Then $T_n \in NBUTA$.

Proof*:* By (1.1), and Pellerey and Petakos ( ), we have that

$[T_n - t \mid T_n > t] \leq_{ttta} \min\{\{X_1 - t \mid X_1 > t\},\ldots,\{X_n - t \mid X_n > t\}\}$

and by Theorem 5.1(a) (Kochar et al. (2002) , we get

$\min\{\{X_1 - t \mid X_1 > t\},\ldots,\{X_n - t \mid X_n > t\}\} \leq_{ttta} T_n = \min\{X_1, X_2,\ldots,X_n\}$

and then the result follows.

Remark 2.1*:* We point out that Corollary 2.1 is a special case of Theorem 3.1 in the next section when $k = 1$.

Finally, because mixtures of some exponential life distributions often belong to the decreasing failure rate (DFR) class (Barlow and Proschan (1981), we conclude that the NBUTA class is not closed under mixtures.

III. BEHAVIOR OF NBUT OF k-OUT-OF- n SYSTEMS

Consider a system of elements with their random lives $X_1, X_2, \ldots, X_n$, respectively. The *k*-out-of-*n* system consists of *n* independent & identically distributed components, and works as long as at least *k* components are working; that is, it works if at most *n-k* components have failed. Thus, the life of a *k*-out-of-*n* system can be characterized by the (*n-k+1*)-th order statistic $X_{n-k+1,n}$. In fact, a series system is an *n*-out-of-*n* system, and a parallel system is a 1-out-of-*n* system. Given that the (*n-k*)-th failure has occurred, the system will fail when the (*n-k+1*)-th failure occurs. Thus, to understand how the aging property of the elements affect the aging procedure of the total life of the whole system, it is of special interest to study the aging procedure of the residual life after the (*n-k*)-th failure.

The residual life of a *k*-out-of-*n*- system, given that the (*n-k*)-th failure occurs at time $t \geq 0$, is represented by the following conditional random variable.

$$RLS_{k,n,t} = \{X_{n-k+1,n} - X_{n-k,n} \mid X_{n-k,n} = t\}$$

and hence the total life of the (*n-k+1*)-th failed element is

$$LS_k = X_{n-k+1,n}$$

In this way, using stochastic comparisons, some authors characterized some aging distributions by the stochastic ordering of the residual life of the *k*-out-of-*n*- system, given that the (*n-k*)-th failure has occurred at different times.

In particular, Langberg et al. (1980) presented the following characterizations.

X is $NBU \Leftrightarrow RLS_{k,n,t} \leq_{st} LS_k$ for all $t \geq 0$,and for any integer k such that $1 \leq k < n$.

Afterward, Belzunce et al. (1999), and Li and Chen (2004) provided additional results on some other stochastic orders and aging notions.

Recently, Li and Zuo (2004) consider the residual life of the system with *i.i.d.* elements. They also pay special attention to the residual life of a 1-out-of-*n* (parallel) system given that the (*n-k*)-th failure occurs at time $t \geq 0$

$$RLP_{k,n,t} = \{X_{n,n} - X_{n-k,n} \mid X_{n-k,n} = t\}$$

and the life span of the longest one within those components should be
$LP_k = X_{n,n}$
They get the following result.
X is $NBUTA \Leftrightarrow RLS_{k,n,t} \le_{icva} LS_k$ for all $t \ge 0$.
Assume the system is composed of *i.i.d.* components. Then
$RLS_{k,n,t} = \min\{(X_1)_t, ..., (X_k)_t\}$ and
$LS_k = \min\{X_1, ..., X_k\}$
and
$RLP_{k,n,t} = \max\{(X_1)_t, ..., (X_k)_t\}$ and
$LP_k = \max\{X_1, ..., X_k\}$,

where $(X_i)_t$, $i = 1, 2, \ldots, k$ are *i.i.d.* copies of X_t .
Our main result of this section presents behaviors of the *NBUTA* class in terms of conditioned residual life.

Theorem 3.1*:* Assume that X is continuous.
(i). If X is NBUTA, then $LS_k \ge_{ttta} RLS_{k,n,t}$, for $t \ge 0$.
(ii). If $LP_k \ge_{ttta} RLP_{k,n,t}$, for $t \ge 0$, then X is NBUTA.
Proof*:* Because X_i is NBUTA
$(X_i)_t \le_{ttta} X_i$ for all $t \ge 0$, $i = 1, 2, \ldots, n$.
(i). In view of Theorem 5.1 (*a*) (Kochar et al. (2002)), it follows that
$\min\{(X_1)_t, ..., (X_k)_t\} \le_{ttta} \min\{X_1, ..., X_k\}$

Thus, we obtain the desired result.
(ii). It can easily be shown that the *TTT*A order has the reversed preservation property under the taking of maximum of *i.i.d.* components (in parallel to Theorem 3.3 (*ii*), *Li and Chan* (2004)),
$\max\{(X_1)_t, ..., (X_k)_t\} \le_{ttta} \max\{X_1, ..., X_k\}$
Hence, X is NBUTA.

IV. TESTING AGAINST ALTERNATIVES

In reliability analysis, testing the ageless notion (the exponential distribution) against positive aging has been wide spread, and of interest for well over four decades. The *NBU* positive aging has been discussed & tested early on through the work of *Hollander & Proschan* [12], and was followed by many authors.
For a recent literature review, and new approaches to this problem, we refer the readers to Ahmad (2001), Ahmad et al. (2001), and Ahmad and Mugdadi (2004), where other classes of positive
aging are also tested & compared. Because the *NBUT* class we presented here includes the *NBU* as a subclass, and hence is an easier to verify aging property, it would be of

interest to test H_o: F is exponential against the alternative H_1: F is *NBUT* , and not exponential. It is possible to propose testing the *NBUT* class along the lines of the *NBU* test proposed by Hollander and Proschan (1972). However, this approach has been found, cf. Ahmad (2001) to have low Pitman efficiencies. Hence we take another new approach that yields better testing in our case than that of the *NBU* one, even though our class is much bigger.
First, we need a measure of departure from H_o in favor of H_1.
The following lemma leads to such a measure.

Lemma 4.1: For X_1 , X_2 , and X_3 which are *i.i.d. & NBUTA*

$$E\{\min(X_1,X_2,X_3)\} \geq \frac{2}{3} E\{\min(X_1,X_2)\}$$

Proof: First observe that if X_1 , X_2 , …, X_k are independent random variables with the same distribution function $F(.)$, and survival function $\bar{F}(.)$, then it is easy to see that

$$E\{\min(X_1,...,X_k)\} = \int_0^\infty \bar{F}^k(u)du$$

According to (1.2), we have that

$$\int_0^1\int_0^\infty \bar{F}^2(t)dF(t) \int_0^{F^{-1}(p)} \bar{F}(x)dxdp - \int_0^1\int_0^\infty \bar{F}(t) \times \int_0^{F_t^{-1}(p)} \bar{F}(x+t)dxdF(t)dp \geq 0$$

But

$$\int_0^1\int_0^\infty \bar{F}^2(t)dF(t) \int_0^{F^{-1}(p)} \bar{F}(x)dxdp = \frac{1}{3}\int_0^1 \int_0^{F^{-1}(p)} \bar{F}(x)dxdp$$

$$= \frac{1}{3}\int_0^\infty\int_0^u \bar{F}(x)dF(u)dx = \frac{1}{3}\int_0^\infty \bar{F}^2(x)dx = \frac{1}{3}E\{\min(X_1,X_2)\} \tag{4.1}$$

Next

$$\int_0^1 \int_0^{F_t^{-1}(p)} \bar{F}(x+t)dxdp = \int_0^\infty\int_0^u \bar{F}(x+t)dxd_uF_t(u)dF(t)$$

$$= -\int_0^\infty\int_0^\infty\int_t^{u+t} \bar{F}(w)dwd_u\bar{F}(u+t)dF(t) = \int_0^\infty\int_0^\infty [\bar{\nu}(t) - \bar{\nu}(u+t)]d_u\bar{F}(u+t)d\bar{F}(t)$$

$$\overset{def}{\equiv} I - II \text{ say}$$

where $\bar{\nu}(x) = \int_x^\infty \bar{F}(u)du$. Now

$$I = \int_0^\infty\int_0^\infty \bar{\nu}(t)d_u\bar{F}(u+t)d\bar{F}(t) = -\int_0^\infty \bar{\nu}(t)\bar{F}(t)d\bar{F}(t)$$

Also

$$II = \int_0^\infty\int_0^\infty \bar{\nu}(u+t)d_u\bar{F}(u+t)d\bar{F}(t) = -\int_0^\infty \bar{\nu}(t)\bar{F}(t)d\bar{F}(t) - \int_0^\infty \bar{F}^2(t)\bar{F}(t)dt$$

$$= -\int_0^\infty \bar{v}(t)\bar{F}(t)d\bar{F}(t) - \int_0^\infty \bar{F}^2(t)dt + \int_0^\infty \bar{F}^3(t)dt$$

Hence

$$I - II = \int_0^\infty \bar{F}^2(t)dt - \int_0^\infty \bar{F}^3(t)dt$$

$$= E\{\min(X_1, X_2)\} - E\{\min(X_1, X_2, X_3)\} \tag{4.2}$$

The result follows from (4.1) & (4.2).

The measure of departure from H_o in favor of H_1 may be taken to be

$$\delta = E\left\{\min(X_1, X_2, X_3) - \frac{2}{3}\min(X_1, X_2)\right\}$$

Note that under H_o, $\delta = 0$ while it is positive under H_1.

To make the test scale-invariant, we take $\Delta = \delta / \mu$.

Let X_1 , X_2 , …, X_n be a random sample from F. We estimate μ by $\bar{X}$, and δ by

$$\hat{\delta} = \frac{1}{n(n-1)(n-2)} \sum_i \sum_{\neq j} \sum_{\neq k} \left\{\min(X_i, X_j, X_k) - \frac{2}{3}\min(X_i, X_j)\right\}$$

Using standard U-statistics theory, we can easily prove the following theorem.

Theorem 4.1*:* As $n \to \infty, \sqrt{n}(\hat{\Delta} - \Delta)$, is asymptotically s-normal with mean 0, and variance $\sigma^2 = \mu^{-2} V(\Psi_1(X_1))$;

where $\Psi_1(X_1) = 6\int_0^{X_1} x\bar{F}(x)dF(x) + 3X_1\bar{F}^2(X_1) - \frac{4}{3}X_1\bar{F}(X_1) - \frac{4}{3}\int_0^{X_1} xdF(x) - \frac{2}{3}\int_0^\infty \bar{F}^2(x)dx$

Under H_o, $\sigma^2 = (2/135)$

Proof: All we need to calculate is the asymptotic variance of $\hat{\delta}$. Set

$$\varphi(X_1, X_2, X_3) = \min(X_1, X_2, X_3) - \frac{2}{3}\min(X_1, X_2)$$

Hence, we easily see that

$$\varphi_1(X_1) = E[\phi(X_1, X_2, X_3) | X_1] = \varphi_2(X_1) = [\phi(X_2, X_1, X_3) | X_1]$$

$$= 2\int_0^{X_1} x\bar{F}(x)dF(x) + X_1\bar{F}^2(X_1) - \frac{1}{3}\int_0^{X_1} xdF(x) - \frac{2}{3}X_1\bar{F}(X_1)$$

Finally

$$\varphi_3(X_1) = E[\phi(X_1, X_2, X_3) | X_1]$$

$$= 2\int_0^{X_1} x\bar{F}(x)dx + X_1\bar{F}^2(X_1) - \frac{2}{3}\int_0^\infty \bar{F}^2(x)dx$$

The result follows by setting $\Psi(X_1) = \sum_{i=1}^{3} \varphi_i(X_1)$

Under H_o, direct calculations give the result.
To carry out the test, calculate $\sqrt{135/2}\hat{\Delta}$, and reject the null hypothesis if this is larger than Z_α from the standard *s*-normal. To assess the goodness of the above procedure, we can use the concept of Pitman Asymptotic Efficacy (PAE) defined as

$$PAE(\Delta) = PAE(\delta) = \frac{\left|\frac{d}{d\theta}\delta_\theta\right|_{\theta\to\theta_0}}{\sigma_0}$$

where , $\delta_\theta = \int \bar{F}_\theta^3 - (2/3)\int \bar{F}_\theta^2$ and θ_0 is the null value of θ.
Let us consider the following three distributions that are in the *NBUT*A class, because they are in the *NBUT* class:

1. *The Weibull Distribution*: $\bar{F}_\theta(x) = e^{-x^\theta}$
2. *The Linear Failure Rate Distribution*: $\bar{F}_\theta(x) = e^{-x-(\theta/2)x^2}$
3. *The Makeham Distribution*: $\bar{F}_\theta(x) = e^{-x-\theta(e^{-x}+x-1)}$.

Note that, in the above three alternatives, one gets the exponential distribution when taking theta equal to 1, 0, and 0, respectively. Calculating the PAE of the above alternatives, we get the values 1.0920 for the Weibull, 0.6203 for the linear failure rate, and 0.292 for the Makeham, respectively. Note also that the NBUT test of Ahmad et al. (2005) has efficacy values of the above three alternatives equal to1.1104, 0.5705, and 0.288, respectively. Thus, our test here is better for the linear failure rate, and for the Makeham, while slightly worse for the Wiebull, even though our class is much larger than that of the NBUT.

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