A note on distinct integer triplets summing to a perfect square in pairs

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Abstract: Let i, j and k be three distinct positive integers (we call it a triplet)such that the sum of every pair in the triplet is a perfect square. For example, (2, 34, 47) is a permissible triplet. Permuting the elements of a triplet among themselves, we get six triplets for each combination. The note raises the non trivial problem of finding a functional relationship between c and n where c is the number of such combination triplets such that i, j, k <=n where n is a positive integer.

Key words: combination triplets; perfect square

Section I is the introduction, section II gives the code used, section III gives the empirical results and section IV is the conclusion and suggestions for future work.

Section I

Introduction: Let i, j and k be three distinct positive integers (triplet) such that the sum of every pair is a perfect square. (6, 19, 30) is the first permissible triplet. Permuting the elements of a triplet among themselves, we get six triplets for each combination. To avoid ambiguity we shall call it a "combination triplet".

Lemma: The set of all combination triplets (with the aforesaid property) is infinite .

Proof: Let (i, j, k) be a combination triplet. Multiply each element of the triplet by a constant positive integer c>1 where *we restrict c to be a perfect square*. We get another triplet (ci, cj, ck) in which ci + cj = c(i+j), cj + ck = c(j+k), ck + ci = c(k+i) are all distinct perfect squares(because c is a perfect square and so are i+j, j+k and k+i)! Hence it is a combination triplet. We again multiply the elements of the new triplet by some constant (>1) which should itself be a perfect square, get another combination triplet, and so on. Since this process of multiplying can be carried out infinitely, we end up with an infinite number of combination triplets, proving the lemma.

Although the proof is trivial it suggests that instead of going for the largest combination triplet (which we would have in case of finiteness) it makes sense to solve the non-trivial problem of finding a functional relationship between c and n where c is the number of combination triplets such that i, j, k <=n where n is a positive integer.

Section II

The following is a QBASIC code running successfully in our system:-

```
    REM  triplet generation
    CLS
    INPUT n
    DIM t1(n), t2(n), t3(n)
    c = 0
    FOR i = 1 TO n
    FOR j = 1 TO n
    FOR k = 1 TO n
    IF i = j OR j = k OR k = i THEN GOTO 20
    s1 = i + j: s2 = j + k: s3 = k + i
    r1 = SQR(s1): r2 = SQR(s2): r3 = SQR(s3)
    IF INT(r1) = r1 AND INT(r2) = r2 AND INT(r3) = r3 THEN GOTO 10 ELSE GOTO 20
10  c = c + 1: t1(c) = i: t2(c) = j: t3(c) = k
20  NEXT k
    NEXT j
    NEXT i
    PRINT "no. of  combination triplets="; c / 6
    PRINT
    FOR i = 1 TO c
    PRINT t1(i), t2(i), t3(i)
    NEXT i
    END
```

Remark: In the code INT(x) gives the integral part of x and SQR(x) the square root of x.
x is a perfect square if and only if INT(SQR(x))=SQR(x).

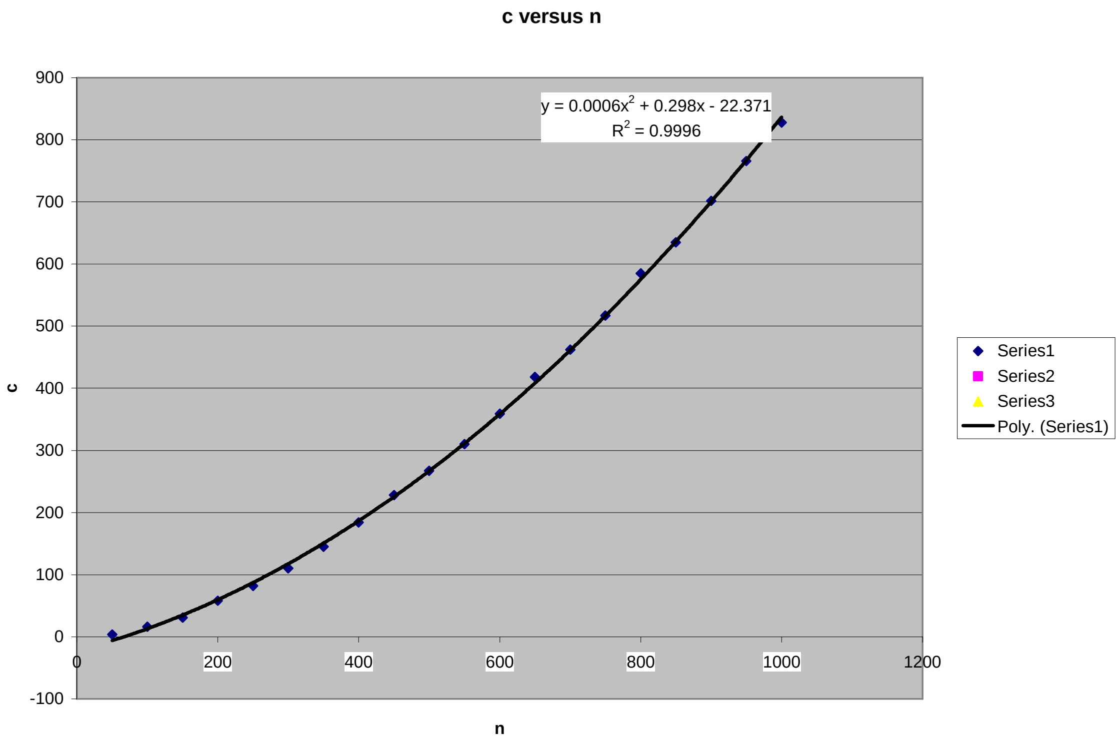
Section III

Here are some interesting results:-

Table 1

n	**50**	**100**	**150**	**200**	**250**	**300**	**350**	**400**	**450**	**500**	**550**	**600**	**650**	**700**	**750**	**800**	**850**	**900**	**950**	**1000**
c	**4**	**16**	**31**	**58**	**82**	**110**	**145**	**184**	**228**	**267**	**310**	**359**	**418**	**462**	**517**	**585**	**635**	**702**	**766**	**828**

Fit to quadratic was found to be adequate in MS Excel 2003 package.
In the quadratic fit, y represents c while x represents n.



Section IV

Conclusion and future work:

Although the exact functional relationship between c and n is non trivial, the model $c = -22.371 + 0.298\,n + 0.0006\,n^2 + \epsilon$ (where ϵ is the error term) seems to be adequate. However we are also trying to establish a theoretical functional relationship between c and n which would be more conclusive. Our future work also includes investigating possible applications which the combination triplets may be put to. We also propose to make the triplet search faster.

References:-

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