THE WEIBULL LENGTH BIASED DISTRIBUTION -PROPERTIES AND ESTIMATION-

By

S. A. Shaban
I.S.S.R Cairo University
drshabanshaban@yahoo.com

Naima Ahmed Boudrissa
I.N.P.S. Algiers University
boudrina2005@yahoo.com

ABSTRACT

The length-biased version of the Weibull distribution known as Weibull length-biased (WLB) distribution is considered, it is shown that it is unimodal throughout examining its shape. Other properties of the distribution were studied such as the moments, and the hazard rate function. It is shown that the hazard function is upside bathtub shaped for values of the shape parameter that are less than unity and increasing otherwise. Bayesian and non Bayesian estimation problems are also considered. a numerical example is introduced for illustration.

Keywords: Length-biased, Hazard rate function, Reliability function, Weibull distribution, Moment method, Maximum likelihood estimates, Bayesian estimates.

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1- INTRODUCTION

The concept of weighted distribution can be traced to Fisher in his paper “The study of effect of methods of ascertainment upon estimation of frequencies” in 1934; while this of length-biased sampling was introduced by Cox 1962 (see Patill 2002). These two concepts find various applications in biomedical area such as family history and disease, survival and intermediate events and latency period of AIDS due to blood transfusion (Gupta and Akman 1995). The study of human families and wildlife populations was the subject of an article developed by Patill and Rao (1978). Patill, et al. (1986) presented a list of the most common forms of the weight function useful in scientific and statistical literature as well as some basic theorems for weighted distributions and size-biased; as special case they arrived at the conclusion that the length-biased version of some mixture of discrete distributions arises as a mixture of the length-biased version of these distributions. Gupta and Tripathi (1990) studied the error made if an ordinary distribution is used instead of the length-biased version, this error was named as type III error or modeling error by Rao (Gupta and Tripathi (1990)). They derived a general form for this error when the random variable under study follows a wide class of discrete distributions known as modified power series distribution. The results were applied to study the family history and diseases using the Poisson and the generalized Poisson distributions. Gupta and Tripathi (1996) studied the weighted version of the bivariate three-parameter logarithmic series distribution, which has applications in many fields such as: ecology, social and behavioral sciences and species abundance studies. They first derived the weighted version of this distribution and gave an explicit form for the probability mass function and the probability generating function in the length-biased case.

Much work was done to characterize relationships between original distributions and their length biased versions. A table for some basic distributions and their length biased forms is given by Patill and Rao (1978) such as Lognormal, Gamma, Pareto, Beta distribution. Khatree (1989) presented a useful result by giving a relationship between the original random variable X and its length-biased version Y, when X is either Inverse Gaussian or Gamma distribution. He showed that the length-biased random variable Y can be written as a linear combination of the original random variable X and a chi-square random variable Z and inversely the original random variable can be characterized through this relationship. Relationships in the context of reliability were treated by several authors such as Patill et al. (1986), Jain et al. (1989), Gupta and Kirmani (1990) and recently by Oulyed and George (2002); In these works the survival function, the failure rate, and the mean residual life function of the length-biased distribution were expressed in relation with these of the original distribution.

Weibull distribution plays an important role in life testing and reliability studies. It was introduced by the Swedish scientist Wallodi Weibull in 1951 (Kapur and Saxena 2001). If X is a random variable having the Weibull distribution, its pdf takes the form:

$$g(x) = \theta\beta x^{\beta-1}\exp(-\theta x^{\beta}) \qquad x \geq 0, \beta > 0, \theta > 0 \tag{1-1}$$

Where: β is the shape parameter and θ scale parameter. The Weibull distribution is very flexible and this is due to its ability in modeling both increasing ($\beta > 1$) and decreasing ($\beta < 1$) failure rates. The Weibull distribution has the following rth moment:

$$E(X^r) = \frac{r}{\beta} \frac{\Gamma(\frac{r}{\beta})}{\theta^{\frac{r}{\beta}}} \tag{1-2}$$

In this pa per the length-biased version of the weibulll distribution is considered which has many applications in forestry and life testing. Let T be a non negative random variable, T is said to have the Weibull length-biased distribution – it will be abbreviated as WLB – if its density function is given by:

$$f(t) = \frac{\beta^2 \theta^{(\frac{1}{\beta}+1)} t^{\beta} e^{-\theta t^{\beta}}}{\Gamma(\frac{1}{\beta})}, \; t > 0, \; \theta, \beta > 0 \tag{1-3}$$

The density (1-3) can be obtained by combining the definition of the length- biased distribution given by:

$$f(t) = \frac{t\, g(t)}{E(t)} \tag{1-4}$$

and the density of the original distribution (1-1). This distribution can be explained as follows: Suppose that the lifetimes of a given sample of items is Weibull and that the items doesn't have the same chance of being selected but each one is selected according to its length or life length then the resulting distribution is not Weibull but Weibull length-biased.

It can be noted that (1-3) is a generalized gamma as defined by Stacy (1962) with parameters $\beta, \eta = \theta^{-\frac{1}{\beta}}, k = \frac{1}{\beta} + 1$.The WLB distribution includes the gamma distribution (β=1) as special case. The reliability function of the WLB distribution is given by:

$$R(t) = 1 - \frac{\beta \gamma(\frac{1}{\beta} + 1, \theta t^{\beta})}{\Gamma(\frac{1}{\beta})} \tag{1-5}$$

The numerator represents the incomplete gamma function defined as:

$$\gamma(a, x) = \int_0^x t^{a-1} e^{-t}\, dt \tag{1-6}$$

The rth moment associated with (1-3) is:

$$E(T^r) = \frac{(r+1)\,\Gamma(\frac{r+1}{\beta})}{\theta^{\frac{r}{\beta}}\,\Gamma(\frac{1}{\beta})} \tag{1-7}$$

The same result can be obtained by using the moments of the original distribution (Weibull) via the following relationship:

$$E(T^r) = \int_0^\infty t^r f(t)\,dt = \frac{1}{E(X)}.\int_0^\infty t^{r+1} g(t)\,dt = \frac{E(X^{r+1})}{E(X)} \tag{1-8}$$

The rest of the paper is organized as follows: the shape of the WLB distribution and its hazard rate function are studied in sections two and three respectively. Bayesian and non Bayesian estimation problems are considered in section four. A numerical example is given in section five to illustrate the above methods.

2- THE SHAPE:

The shape of the density function given in (1-3) can be clarified by studying this function defined over the positive real line [0, ∞] and the behavior of its derivative as follows:

2-1- Limits and derivatives of the function

We have:

1- $\lim_{t\to 0} f(t) = \frac{\beta^2 \theta^{\frac{1}{\beta}+1}}{\Gamma(\frac{1}{\beta})} \lim_{t\to 0}\ t^\beta \exp(-\theta t^\beta)$

but: $\lim_{t\to 0} t^\beta = 0 \quad and \quad \lim_{t\to 0} \exp(-\theta t^\beta) = 1$

it results that:

$$\lim_{t\to 0} f(t) = 0 \tag{2-1}$$

2- $\lim_{t\to \infty} f(t) = \frac{\beta^2 \theta^{\frac{1}{\beta}+1}}{\Gamma(\frac{1}{\beta})} \lim_{t\to \infty}\ t^\beta \exp(-\theta t^\beta)$

If we put $z = \theta t^\beta$, the above limit can be rewritten as:

$$\lim_{t\to\infty} f(t) = \frac{\beta^2 \theta^{\frac{1}{\beta}}}{\Gamma(\frac{1}{\beta})} \lim_{z\to\infty} z\exp(-z)$$

but : $\lim_{t\to\infty} z\exp(-z) = 0$, from this follows that

$$\lim_{t\to\infty} f(t) = 0 \qquad (2\text{-}2)$$

Secondly we study the derivative, since the function f(t) and its logarithm are maximized at the same point and in order to simplify calculations we will take the derivative of the logarithm of the function f(t) given by

$$\ln(f(t)) = 2\ln\beta + \beta\ln t - \theta t^{\beta} - \ln\Gamma(\frac{1}{\beta}) + (\frac{1}{\beta}+1)\ln\theta \qquad (2\text{-}3)$$

Taking the derivative of this function with respect to t yields:

$$\frac{\partial}{\partial t}\ln(f(t)) = \beta\,(\frac{1}{t} - \theta t^{\beta-1})$$

Equating this derivative to zero gives:

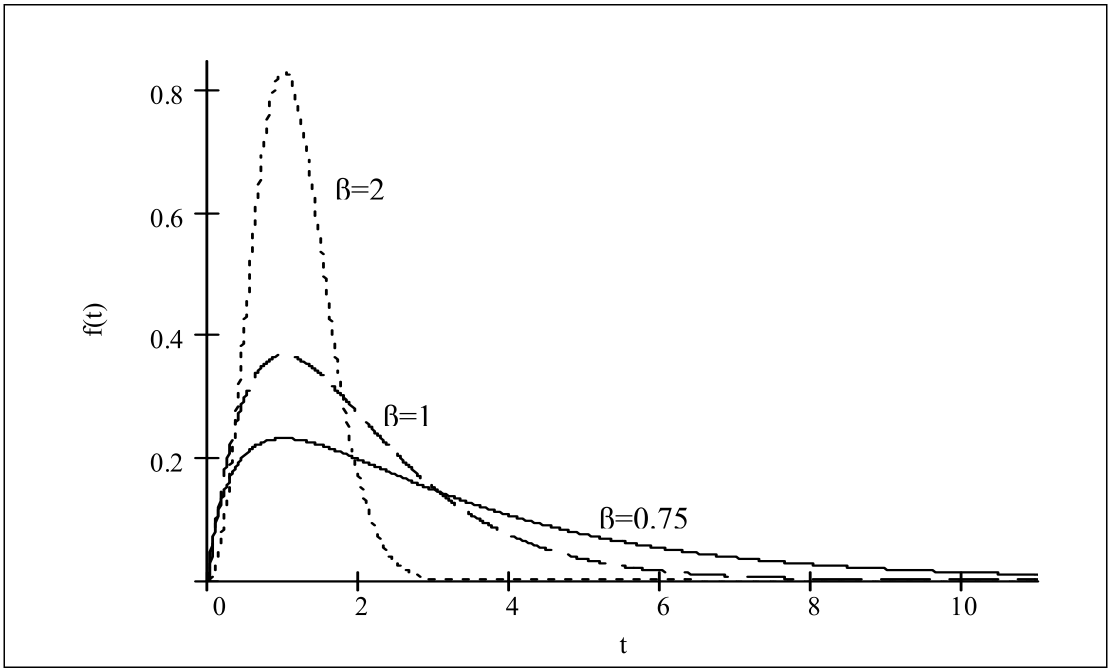
$$t_0 = \theta_0^{-\frac{1}{\beta_0}} \qquad (2\text{-}4)$$

Then the derivative is equal zero at t_0, negative for values of t that exceed t_0 and positive otherwise. To verify if the point (θ_0 , β_0) is a maximum or minimum, the second derivative of f(t) with respect to t is derived which is equal to:

$$\frac{\partial^2}{\partial t^2}\ln(f_2(t)) = -\frac{\beta}{t^2}(1 + \theta(\beta-1)t^{\beta}) \qquad (2\text{-}5)$$

This quantity is negative for all values of t. From the above results the function f (t) increases it takes its maximum at t_0 then it decreases again. The following figure illustrates some of the possible shapes of the density f(t) for specified values of β. The scale parameter θ was taken to be unity since it doesn't influence the shape of this function.

Figure 1: The density function of the WLB distribution



The shape of the distribution can be studied in more details using the following two coefficients.

2-2- Coefficient of skewness:

Denote it by Sk, the coefficient of skewness enables us to know if the distribution under study is symmetric or not, it is defined by:

$$Sk = \frac{\mu_3}{\sigma^3} \qquad (2\text{-}6)$$

Where: μ_3 is the third moment about the mean and σ is the standard deviation of the distribution. The skewness is zero for symmetrical distributions, positive for skewed right distributions, and negative if the distribution is skewed to the left (Frank and Althoen, 1994) this means that the sign of the coefficient indicates the direction of the skew. From equation (2-1) for r=1, 2, 3 and equation (2-3). Replacing them in equation (3-6) gives:

$$SK = \frac{4\,\Gamma^2(\frac{1}{\beta})\,\Gamma(\frac{4}{\beta}) - 18\,\Gamma(\frac{3}{\beta})\,\Gamma(\frac{2}{\beta})\,\Gamma(\frac{1}{\beta}) + 16\,\Gamma^3(\frac{2}{\beta})}{\left(3\,\Gamma(\frac{3}{\beta})\,\Gamma(\frac{1}{\beta}) - 4\,\Gamma^2(\frac{2}{\beta})\right)^{\frac{3}{2}}} \qquad (2\text{-}7)$$

It can be noted from (3-7) that the coefficient of skewness of the WLB distribution doesn't depend on the scale parameter θ, and it is function of the shape parameter β only then we can write it as $Sk(\beta)$. Numerical investigation of $Sk(\beta)$ indicates that the WLB distribution is symmetric for β=3.448 -at this point the mean is equal to the median -, positively skewed with a tail to the right for values (β<3.448), and negatively skewed with a tail to the left for (β>3.448). In practice more attention is given to the first case (i.e. for β<3.448).

2-3- Coefficient of kurtosis:

Denote it by Kur, the coefficient of kurtosis measures the flatness of the top and it is defined by:

$$Kur = \frac{\mu_4}{\sigma^4} - 3 \tag{2-8}$$

The kurtosis is equal zero for the normal distribution, positive for the more tall and slim curves than the normal one in the neighborhood of the mode, in this case the distribution is said to be leptokurtic. It is negative for platykurtic distributions (i.e. flatter than the normal distribution). Using the moments from (2-1) and the variance from (2-3) the coefficient of kurtosis for the WLBD is given by:

$$Kur = \frac{5\Gamma^3(\frac{1}{\beta})\Gamma(\frac{5}{\beta}) - 32\Gamma(\frac{4}{\beta})\Gamma(\frac{2}{\beta})\Gamma^2(\frac{1}{\beta})}{\left(3\Gamma(\frac{3}{\beta})\Gamma(\frac{1}{\beta}) - 4\Gamma^2(\frac{2}{\beta})\right)^2}$$

$$+ \frac{72\Gamma(\frac{3}{\beta})\Gamma^2(\frac{2}{\beta})\Gamma(\frac{1}{\beta}) - 48\Gamma^4(\frac{2}{\beta})}{\left(3\Gamma(\frac{3}{\beta})\Gamma(\frac{1}{\beta}) - 4\Gamma^2(\frac{2}{\beta})\right)^2} - 3 \tag{2-9}$$

Kur is also a function on the parameter β only and it can be written as $Kur = Kur(\beta)$, the Kurtosis is positive for values of the shape parameter that are (β ≤ 2.164 or β > 5.455) then the WLB distribution's curve is leptokurtic (thin), it is platycurtic for (2.164 ≤ β < 5.455) since the coefficient is negative for this case. It is near zero in the neighborhood of each of the two points.

3-HAZARD RATE FUNCTION

The hazard rate function is defined by the ratio $\left(f(t)/(1-F(t))\right)$, it takes the form:

$$h_2(t)=\frac{\beta^2\,\theta^{\frac{1}{\beta}+1}\,t^{\beta}e^{-\theta t^{\beta}}}{\left(\Gamma(\frac{1}{\beta})-\beta\gamma(\frac{1}{\beta}+1,\theta t^{\beta})\right)} \tag{3-1}$$

In order to study the behavior of this hazard rate we apply results of Glaser (1980) given in the form of lemma (3-1).

Lemma (3-1):

Let T be a continuous random variable with twice differentiable density function $f(t)$. Define the quantity $\eta(t)=-\frac{f'(t)}{f(t)}$, where $f'(t)$ denote the first derivative of the density function with respect to t. suppose that the first derivative of $\eta(t)$ -named $\eta'(t)$ - exists. Glaser gave the following results (for more details see Glaser (1980)).

1- If $\eta'(t)<0$, for all t>0, then the hazard rate is monotonically decreasing failure rate (DFR).

2- If $\eta'(t)>0$, for all t>0, then the hazard rate is monotonically increasing failure rate (IFR).

3- If there exists t_0, such that $\eta'(t)>0$ for all ($0 < t < t_0$); $\eta'(t_0)=0$ and $\eta'(t)<0$ for all ($t>t_0$). In addition to that $\lim_{t\to 0} f(t)=0$; then the hazard rate is upside down bathtub shaped (UBT).

4- If there exists t_0, such that $\eta'(t)<0$ for all $0<t<t_0$; $\eta'(t_0)=0$ and $\eta'(t)>0$ for all $t>t_0$. Adding to that $\lim_{t\to 0} f(t)=\infty$ .it consequences that the hazard rate is bathtub shaped (BT).

For WLB distribution we begin by computing the quantity $\eta(t)$; by first taking the derivative of the density function given in (1-3) with respect to t which is given by:

$$f'(t)=\frac{\partial f_2(t)}{\partial t}=\frac{\beta}{t}f(t)(1-\theta t^{\beta}) \tag{3-2}$$

Dividing both sides of the equation (3-2) by the measure $(-f(t))$ we obtain: $\eta(t)=\frac{\beta}{t}\left(\theta t^{\beta}-1\right)$, taking its derivative with respect to t yields:

$$\eta'(t)=\frac{\beta}{t^2}\left(1+(\beta-1)\theta t^{\beta}\right) \tag{3-3}$$

According to the values of the shape parameter β:

1- For $\beta < 1$, it is easily seen that the third part of the lemma follows; where t_0 is solution of $\eta'(t_0) = 0 \Rightarrow t_0 = (\theta(1-\beta))^{-\frac{1}{\beta}}$. It results that the hazard rate is UBT shaped.

2- For $\beta = 1, \eta'(t) = \frac{1}{t^2}$, which is strictly positive function for all values of t. It results from the lemma (4-1) that $h(t)$ is IFR, in this case also the WLBD reduces to gamma distribution with parameter ($k = \frac{1}{\beta} + 1 = 2$) with an increasing hazard rate.

3- For $\beta > 1, \eta'(t) > 0$ for all t, then the hazard rate is monotonically increasing (IFR); this agrees with the theorem given in Gupta and Kirmani (1990) which indicated that the length-biased version preserves the IFR property of the original random variable.

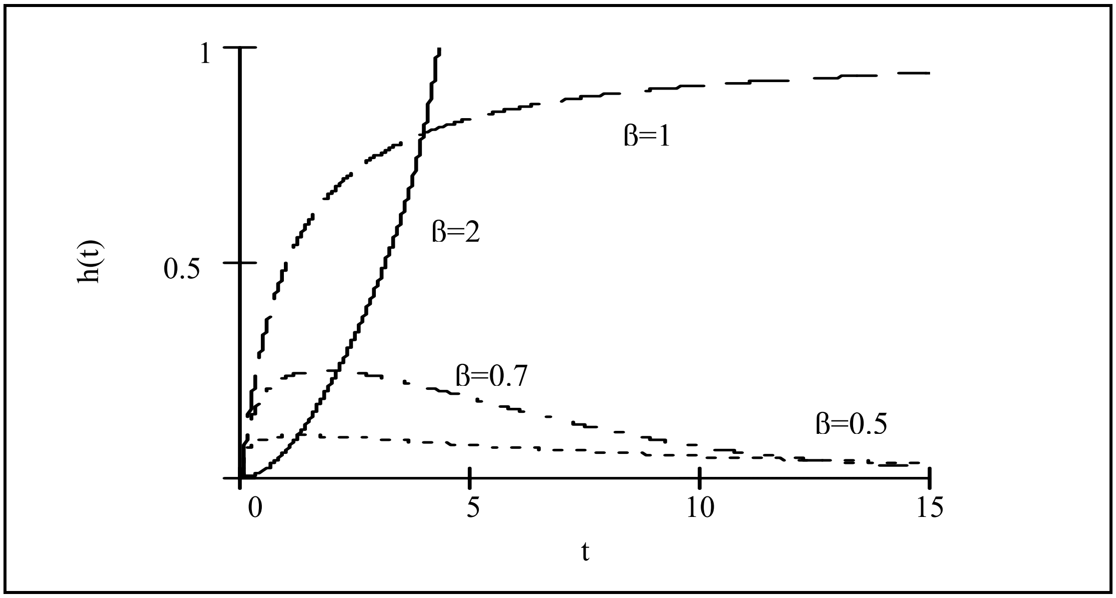
These results can be summarized through the following theorem:

Theorem (3-1):

let T be a non negative random variable having the Weibull length-biased distribution; then its hazard rate $h(t)$ is IFR for values of the shape parameter that are greater or equal one ($\beta \geq 1$), and UBT otherwise –it means for ($0 < \beta < 1$).

The shapes of the hazard rate of the WLB distribution for special values of the shape parameter β are illustrated in figure (2), the scale parameter θ was taken to be the unity since it doesn't influence the shape of the hazard rat

Figure 2: The Hazard rate of the WLB distribution for given values of β



4-ESTIMATION

In this section estimates of the two parameters of the WLB distribution and the reliability function are obtained by the method of moment, maximum likelihood method, and Bayesian and approximate Bayesian method -using Lindley's expansion- assuming independent non-informative prior for each parameter.

4-1- Moment estimates

This method follows by equating the population moments from (1-7) to the sample moments this yields the following system of two equations:

$$\frac{1}{n}\sum_{i=1}^{n} t_i = \frac{2\Gamma(\frac{2}{\beta^*})}{(\theta^*)^{\frac{1}{\beta^*}}\Gamma(\frac{1}{\beta^*})} \qquad (4\text{-}1\text{-}a)$$

$$\frac{1}{n}\sum_{i=1}^{n} t_i^2 = \frac{3\Gamma(\frac{3}{\beta^*})}{(\theta^*)^{\frac{2}{\beta^*}}\Gamma(\frac{1}{\beta^*})} \qquad (4\text{-}1\text{-}b)$$

Solving this system will yield θ^*, β^* the moment's estimates of θ and β respectively. These estimates are generally used as initial values for the maximum likelihood method when no closed form exists for the MLE and the normal equations needs to be solved iteratively. Replacing these estimates in the formula of the reliability (1-5) yields:

$$R_2^*(t) = 1 - \frac{\gamma(\frac{1}{\beta^*}+1, \theta^* t^{\beta^*})}{\Gamma(\frac{1}{\beta^*}+1)} \tag{4-2}$$

This estimate can be called the moment estimate of the reliability function.

4-2- Maximum likelihood estimates (MLE):

Suppose that a sample was drawn from (1-3), then the logarithm of the likelihood is given by:

$$l = 2n\ln\beta + \beta\sum_{i=1}^{n}\ln(t_i) - \theta\sum_{i=1}^{n}(t_i)^{\beta} + n(\frac{1}{\beta}+1)\ln\theta - n\ln(\Gamma(\frac{1}{\beta})) \tag{4-3}$$

Differentiating (4-3) with respect to θ and β in turns and equating the derivatives to zero, we get the following normal equations:

$$-\sum_{i=1}^{n}(t_i^{\hat{\beta}}) + \frac{n(\frac{1}{\hat{\beta}}+1)}{\hat{\theta}} = 0 \tag{4-4-a}$$

$$\frac{2n}{\hat{\beta}} + \sum_{i=1}^{n}\ln(t_i) - \hat{\theta}\sum_{i=1}^{n}\ln(t_i)(t_i^{\hat{\beta}}) + \frac{n}{\hat{\beta}^2}\left(\Psi(\frac{1}{\hat{\beta}}) - \ln\hat{\theta}\right) = 0 \tag{4-4-b}$$

Where the Psi-function $\Psi(a)$ is defined as the derivative of the logarithm of the gamma function with respect to a. (see the Handbook of Mathematical Functions 1970, page 259)

$$\Psi(a) = \frac{\partial}{\partial a}(\ln(\Gamma(a))) = \frac{\Gamma'(a)}{\Gamma(a)}, \quad a>0 \tag{4-5}$$

The Psi-function is known as digamma function which can be approximated by:

$$\Psi(a) \sim \ln(a) - \frac{1}{2a} - \frac{1}{12a^2} + \frac{1}{120a^4} - \frac{1}{252a^6} + \ldots\ldots\ldots \tag{4-6}$$

The system (4-4-a), (4-4-b) can be reduced to only one equation by extracting $\hat{\theta}$ from the first equation:

$$\hat{\theta} = \frac{n(\frac{1}{\hat{\beta}}+1)}{\sum_{i=1}^{n}(t_i^{\hat{\beta}})} \tag{4-7}$$

And replace it in the second one i.e. equation (4-4-b), we obtain:

$$\frac{2n}{\hat{\beta}}+\sum_{i=1}^{n}\ln(t_i)-\frac{n(\frac{1}{\hat{\beta}}+1)\left(\sum_{i=1}^{n}\ln(t_i)(t_i^{\hat{\beta}})\right)}{\sum_{i=1}^{n}(t_i^{\hat{\beta}})}+\frac{n}{\hat{\beta}^2}\left(\ln(\sum_{i=1}^{n}t_i^{\hat{\chi}})-\ln(n)-\ln(\frac{1}{\hat{\beta}}+1)+\Psi(\frac{1}{\hat{\beta}})\right)=0 \tag{4-8}$$

This nonlinear equation doesn't seem to have a closed form solution and must be solved iteratively to obtain the estimate of the shape parameter which will be replaced in (4-7) to get the MLE of the scale parameter θ. Or the system of the two equations can be solved simultaneously. The asymptotic expected variance-covariance matrix of the estimates can be obtained by inverting the information matrix (see the Appendix) with elements that are negatives of the expected values of the second derivatives of the likelihood function with respect to the parameters θ and β:

$$I_{11}=E(-\frac{\partial^2 l}{\partial\theta^2})=\frac{n(\frac{1}{\beta}+1)}{\theta^2} \tag{4-9-a}$$

$$I_{12}=I_{21}=E(-\frac{\partial^2 l}{\partial\theta\partial\beta})=\frac{n}{\beta^2\theta}\left\{(\beta+1)\left(\Psi(\frac{1}{\beta})+\frac{\beta(\beta+2)}{(\beta+1)}-\ln(\theta)\right)+1\right\} \tag{4-9-b}$$

$$I_{22}=E(-\frac{\partial^2 l}{\partial\beta^2})=\frac{2n}{\beta^2}+\frac{n}{\beta^3}\left\{(\beta+1)\left[\left(\Psi(\frac{1}{\beta})+\frac{\beta(\beta+2)}{(\beta+1)}-\ln(\theta)\right)^2+\xi(2,\frac{1}{\beta}+1)\right]+\frac{\Psi'(\frac{1}{\beta})}{\beta}-2\ln(\theta)+2\Psi(\frac{1}{\beta})\right\} \tag{4-9-c}$$

Where $\Psi'(.)$ is the derivative of the digamma function, it can be approximated by taking the derivative of (4-6), and $\xi(z,a)$ is Rieman's zeta function (see Gradshteyn and Ryzhik (1965), pages:1072-1073) it is defined by:

$$\xi(z,a)=\frac{1}{\Gamma(z)}\int_0^\infty\frac{t^{z-1}\exp(-at)}{1-\exp(-t)}dt=\sum_{i=0}^{\infty}\frac{1}{(a+i)^z} \tag{4-10}$$

By replacing z and a by their values or formula we get:

$$\xi(2,\frac{1}{\beta}+1)=\int_0^\infty\frac{t\exp(-t(\frac{1}{\beta}+1))}{1-\exp(-t)}dt=\sum_{i=0}^{\infty}\frac{1}{(\frac{1}{\beta}+i+1)^2} \tag{4-11}$$

Then the variance-covariance matrix of the estimates of the parameters can be obtained by inverting the information matrix as follows:

$$Var\ (\theta\ ,\beta) = \begin{bmatrix} I_{11} & I_{12} \\ I_{21} & I_{22} \end{bmatrix}^{-1} = \begin{bmatrix} Var(\theta) & Cov(\theta\ ,\beta) \\ Cov(\theta\ ,\beta) & Var(\beta) \end{bmatrix} \qquad (4\text{-}12)$$

An observed variance-covariance matrix can be obtained also by replacing the MLE in the information matrix and inverting it without taking expectations.

4-2-1- Special cases:

If one of the two parameters is known we have the following results:

1- If the shape parameter β is known, the MLE of the scale parameter θ is given by:

$$\hat{\theta} = \frac{n(\frac{1}{\beta}+1)}{\sum_{i=1}^{n} t_i^{\beta}} \qquad (4\text{-}13\text{-}a)$$

With variance

$$Var(\hat{\theta}) = \frac{\theta^2 \beta}{n(1+\beta)} \qquad (4\text{-}13\text{-}b)$$

2- If the scale parameter θ is known the MLE of the shape parameter can be obtained by solving equation (4-4-b) after replacing $\hat{\theta}$ by θ and its variance is obtained by inverting (4-9-c) and replacing $\hat{\theta}$ by θ too.

5-2-2- Maximum likelihood estimate of the reliability function

The reliability function can be regarded as a parameter and it needs to be estimated. Using the invariance property of the maximum likelihood method, the MLE $\hat{R}$ of the reliability R can be obtained by replacing $\hat{\theta}\ and\ \hat{\beta}$, the maximum likelihood estimates of $\theta\ and\ \beta$ in the formula (1-5) it is given by

$$\hat{R} = \hat{R}(t) = 1 - \frac{\hat{\beta}\gamma(\frac{1}{\hat{\beta}}+1, \hat{\theta}t^{\beta})}{\Gamma(\frac{1}{\hat{\beta}})} \qquad (4\text{-}14)$$

Using Taylor expansion of order one about the parameter estimates of $\hat{R}$ we can write:

$$\hat{R} = R + \frac{\partial R}{\partial \theta}(\hat{\theta}-\theta) + \frac{\partial R}{\partial \beta}(\hat{\beta}-\beta)$$

By taking the expectation of the above formula and from the properties of the MLE, it results that $\hat{R}$ is asymptotically unbiased estimate of R with variance:

$$V(\hat{R}) = (\frac{\partial R}{\partial \theta})^2 V(\hat{\theta}) + (\frac{\partial R}{\partial \beta})^2 V(\hat{\beta}) + 2\frac{\partial R}{\partial \theta}\frac{\partial R}{\partial \beta} COV(\hat{\theta},\hat{\beta}) \qquad (4\text{-}15)$$

Where the variances and covariances of the maximum likelihood estimates of the parameters were given in the matrix (4-12).

4-3- Bayes estimates:

Suppose that a little information is available about the parameters, and then the appropriate prior for this case assuming independence is:

$$\pi(\theta,\beta) \propto \frac{1}{\theta\,\beta} \qquad (4\text{-}16)$$

$\propto$: being the sign of proportionality. Using Bayes theorem which combines the likelihood function with the prior given in (4-16), we obtain the following joint posterior:

$$\pi(\theta,\beta/T) \propto \frac{\beta^{2n-1}\theta^{n(\frac{1}{\beta}+1)-1}}{\Gamma^n(\frac{1}{\beta})} \exp\left\{-\theta\sum_{i=1}^{n} t_i^{\beta} + \beta\sum_{i=1}^{n}\ln(t_i)\right\} \qquad (4\text{-}17)$$

Using a squared error loss function the Bayes estimates of any function of parameters is its posterior expectation given by:

$$E(u(\eta)/t) = \frac{\int_{\Omega} u(\eta)\pi(\eta/t)\,d\eta}{\int_{\Omega}\pi(\eta/t)\,d\eta} \qquad (4\text{-}18)$$

Where η is a parameter which is in this case $\eta = (\beta,\theta)$ or it is equal to one of the two parameters if the other one is known, and Ω is the parameter space. When the analytical method is not attractable we refer to numerical integration to obtain the Bayes estimates.

4-3-1- Bayes estimate of the scale parameter θ:

Putting $u(\eta) = u(\beta,\theta) = \theta$ in (4-18) and using the posterior in (4-17) we get the Bayes estimate of the scale parameter by a ratio of two integrals this means:

$$\tilde{\theta} = E(\theta/T) = \frac{\int_0^{\infty}\int_0^{\infty}\theta\,\pi(\theta,\beta/T)\,d\beta\,d\theta}{\int_0^{\infty}\int_0^{\infty}\pi(\theta,\beta/T)\,d\beta\,d\theta} = \frac{C_2}{C_1} \qquad (4\text{-}19\text{-}a)$$

Where C_1 is the normalizing constant; it is given by:

$$C_1 = \int_0^\infty \frac{\beta^{2n-1}\Gamma\left(n(\frac{1}{\beta}+1)\right)}{\Gamma^n(\frac{1}{\beta})} \exp\left(\beta\sum_{i=1}^n \ln(t_i) - n(\frac{1}{\beta}+1)\ln(\sum_{i=1}^n t_i^\beta)\right)d\beta \tag{4-19-b}$$

And

$$C_2 = \int_0^\infty \frac{\beta^{2n-1}\Gamma\left(n(\frac{1}{\beta}+1)+1\right)}{\Gamma^n(\frac{1}{\beta})} \exp\left(\beta\sum_{i=1}^n \ln t_i - (n(\frac{1}{\beta}+1)+1)\ln(\sum_{i=1}^n t_i^\beta)\right)d\beta \tag{4-19-c}$$

The variance of the Bayes estimate of scale parameter θ can be obtained by applying the following formula:

$$Var(\tilde{\theta}) = E(\theta^2/T) - E^2(\theta/T) = \frac{C_3}{C_1} - \left(\frac{C_2}{C_1}\right)^2 \tag{5-19-d}$$

The second posterior moment can be obtained by setting $u(\eta) = u(\beta,\theta) = \theta$ in (4-18) and using the posterior in (4-17) this yield:

$$E(\theta^2/T) = \frac{\int_0^\infty\int_0^\infty \theta^2\, \pi(\theta,\beta/T)d\beta\, d\theta}{\int_0^\infty\int_0^\infty \pi(\theta,\beta/T)d\beta\, d\theta} = \frac{C_3}{C_1} \tag{4-19-e}$$

$$C_3 = \int_0^\infty \frac{\beta^{2n-1}\Gamma\left(n(\frac{1}{\beta}+1)+2\right)}{\Gamma^n(\frac{1}{\beta})} \exp\left(\beta\sum_{i=1}^n \ln t_i - (n(\frac{1}{\beta}+1)+2)\ln(\sum_{i=1}^n t_i^\beta)\right)d\beta \tag{4-19-f}$$

No closed form solutions have been found for the integrals in (4-19-b), (4-19-c) and (4-19-f) when both parameters are unknown and numerical integration is necessary to evaluate them. If we suppose that the shape parameter β is known the integrals in (4-19-a) and (4-19-b) will admit closed form solutions, the Bayes estimate of the scale parameter and its variance are identical to those of the maximum likelihood method given by (4-13-a), (4-13-b).

4-3-2- The shape parameter β:

Setting $u(\eta) = u(\beta,\theta) = \beta$ in (4-18) and using the posterior in (4-17) we get the Bayes estimate of the shape parameter by a ratio of two integrals this means:

$$\tilde{\beta} = E(\beta/T) = \frac{\int_0^\infty\int_0^\infty \beta\, \pi(\theta,\beta/T)d\theta\, d\beta}{\int_0^\infty\int_0^\infty \pi(\theta,\beta/T)d\theta\, d\beta} = \frac{C_4}{C_1} \tag{4-20-a}$$

The denominator C_1 was given by equation (4-19-b) given in and the numerator is given by:

$$C_4 = \int_0^\infty \frac{\beta^{2n}\,\Gamma\left(n(\frac{1}{\beta}+1)\right)}{\Gamma^n(\frac{1}{\beta})} \exp\left(\beta\sum_{i=1}^{n}\ln t_i - n(\frac{1}{\beta}+1)\ln(\sum_{i=1}^{n} t_i^{\beta})\right) d\beta \tag{4-20-b}$$

And

$$E(\beta^2/T) = \frac{\int_0^\infty\int_0^\infty \beta^2\,\pi(\theta,\beta/T)\,d\theta\,d\beta}{\int_0^\infty\int_0^\infty \pi(\theta,\beta/T)\,d\theta\,d\beta} = \frac{C_5}{C_1} \tag{4-20-c}$$

$$C_5 = \int_0^\infty \frac{\beta^{2n+1}\,\Gamma(n(\frac{1}{\beta}+1))}{\Gamma^n(\frac{1}{\beta})} \exp\left(\beta\sum_{i=1}^{n}\ln t_i - n(\frac{1}{\beta}+1)\ln(\sum_{i=1}^{n} t_i^{\beta})\right) d\beta \tag{4-20-d}$$

The variance of the Bayes estimate of the shape parameter β is then given by:

$$Var(\tilde{\beta}) = E(\beta^2/T) - E^2(\beta/T) = \frac{C_5}{C_1} - \left(\frac{C_4}{C_1}\right)^2 \tag{4-20-e}$$

When the scale parameter θ is known, the integrals in (4-19-b), (4-20-b), (4-20-d) don't have a closed form expression and will reduce to:

$$C_1 = \int_0^\infty \frac{\beta^{2n-1}\theta^{n(\frac{1}{\beta}+1)}}{\Gamma^n(\frac{1}{\beta})} \exp\left\{-\theta\sum_{i=1}^{n} t_i^{\beta} + \beta\sum_{i=1}^{n}\ln(t_i)\right\} d\beta \tag{4-21-a}$$

$$C_4 = \int_0^\infty \frac{\beta^{2n}\theta^{n(\frac{1}{\beta}+1)}}{\Gamma^n(\frac{1}{\beta})} \exp\left\{-\theta\sum_{i=1}^{n} t_i^{\beta} + \beta\sum_{i=1}^{n}\ln(t_i)\right\} d\beta \tag{4-21-b}$$

$$C_5 = \int_0^\infty \frac{\beta^{2n+1}\theta^{n(\frac{1}{\beta}+1)}}{\Gamma^n(\frac{1}{\beta})} \exp\left\{-\theta\sum_{i=1}^{n} t_i^{\beta} + \beta\sum_{i=1}^{n}\ln(t_i)\right\} d\beta \tag{4-21-c}$$

Then in both cases (θ known or unknown) numerical integration is necessary to evaluate these integrals.

4-3-3- The reliability function:

Putting $u(\eta) = u(\beta,\theta) = R(t)$ in (4-18) and using the posterior in (4-17) we get the Bayes estimate of the shape parameter by a ratio of two integrals this means:

$$\tilde{R}(t) = E(R(t)/T) = \frac{\int_0^\infty\int_0^\infty R(t)\,\pi(\theta,\beta/T)\,d\theta\,d\beta}{\int_0^\infty\int_0^\infty \pi(\theta,\beta/T)\,d\theta\,d\beta} = \frac{C_6}{C_1} \qquad \text{(4-22-a)}$$

With:

$$C_6 = \int_0^\infty\int_0^\infty \frac{\beta^{2n-1}\left(\Gamma(\frac{1}{\beta}) - \beta\gamma(\frac{1}{\beta}+1,\theta t^\beta)\right)\theta^{n(\frac{1}{\beta}+1)-1}\exp\left(-\theta\sum_{i=1}^n t_i^\beta + \beta\sum_{i=1}^n \ln(t_i)\right)}{\left(\Gamma(\frac{1}{\beta})\right)^{n+1}}\,d\theta\,d\beta \qquad \text{(4-22-b)}$$

And its variance can be obtained from:

$$Var(\tilde{R}) = E(R^2/T) - E^2(R/T) \qquad \text{(4-22-c)}$$

Where: $$E(R^2(t)/T) = \frac{\int_0^\infty\int_0^\infty R^2(t)\pi(\theta,\beta/T)\,d\theta\,d\beta}{\int_0^\infty\int_0^\infty \pi(\theta,\beta/T)\,d\theta\,d\beta} = \frac{C_7}{C_1} \qquad \text{(4-22-d)}$$

$$C_7 = \int_0^\infty\int_0^\infty \frac{\beta^{2n-1}\left(\Gamma(\frac{1}{\beta}) - \beta\gamma(\frac{1}{\beta}+1,\theta t^\beta)\right)^2 \theta^{n(\frac{1}{\beta}+1)-1}}{\left(\Gamma(\frac{1}{\beta})\right)^{n+2}}\exp\left(-\theta\sum_{i=1}^n t_i^\beta + \beta\sum_{i=1}^n \ln(t_i)\right)d\theta d\beta \qquad \text{(4-22-e)}$$

No closed form solutions exist for the integrals in (4-22-a) and (4-22-b) even if one of the two parameters is known, these integrals can be computed via numerical integration.

4-4- Approximate Bayes estimates:

When the integrals occurring in Bayesian analysis don't admit closed form solution we refer to numerical integration to find a solution as it was suggested in the precedent section. Lindley (1980) gave an alternative method to approximate the integrals that occur in Bayesian statistics. The form of ratio of integrals considered by Lindley (1980) is as given bellow:

$$\frac{\int w(\eta)\exp(l(\eta))\,d\eta}{\int v(\eta)\exp(l(\eta))\,d\eta} \qquad \text{(4-23)}$$

Where: $\eta = (\eta_1, \eta_2, \cdots, \eta_m)$ is the parameter, $l(\eta)$ is the logarithm of the likelihood function and $w(.), v(.)$ are arbitrary functions of η. Let $v(\eta) = \pi(\eta)$ the prior density of the parameter η, $w(\eta) = u(\eta)\pi(\eta)$ and $\rho(\eta) = \ln(\pi(\eta))$, the ratio in (4-25) will be the posterior expectation of the function $u(\eta)$ under squared error loss function and we write:

$$E(u(\eta)/t) = \frac{\int u(\eta)\exp(l(\eta)+\rho(\eta))\,d\eta}{\int \exp(l(\eta)+\rho(\eta))\,d\eta} \tag{4-24}$$

This ratio is equivalent to the ratio given in (4-18); The basic idea to evaluate it is to expand on Taylor series the functions involved in it about the maximum likelihood $\hat{\eta}$ of η, this lead to the following formula, where the first term omitted is $O(n^{-2})$:

$$E(u(\eta)/t) \sim u + \frac{1}{2}\sum_{i,j}(u_{ij} + 2u_i\rho_j)\sigma_{ij} + \frac{1}{2}\sum_{i,j,k} l_{ijk}\,u_l\,\sigma_{ij}\,\sigma_{kl} \tag{4-25}$$

Where each suffix denotes differentiation once with respect to the variable having that suffix; this means:

$l_{ijk} = \dfrac{\partial^3 l(\eta)}{\partial\eta_i\,\partial\eta_j\,\partial\eta_k}$, $u_{ij} = \dfrac{\partial^2 u(\eta)}{\partial\eta_i\,\partial\eta_j}$, $u_i = \dfrac{\partial u(\eta)}{\partial\eta_i}$, $\rho_i = \dfrac{\partial\rho(\eta)}{\partial\eta}$, etc. σ_{ij} is the (i,j) element of the variance covariance-matrix with elements that are the inverse of negatives of the second derivatives of the log likelihood with respect to the parameters. All the quantities in (4-25) are to be evaluated at the MLE of θ and the summation run over all suffixes from one to m (the dimensionality of θ). Lindley (1980) gave the one–parameter and two-parameter version of (4-25) as follows:

For the one- parameter case:

$$E(u(\eta)/t) \sim u(\eta) + \frac{1}{2}(u_2 + 2u_1\rho_1)\sigma^2 + \frac{1}{2}l_3 u_1\sigma^4 \tag{4-26-a}$$

Where:

$$u_1 = \frac{\partial u(\eta)}{\partial\eta},\; u_2 = \frac{\partial^2 u(\eta)}{\partial\eta^2},\; \rho_1 = \frac{\partial\rho}{\partial\eta},\; l_2 = \frac{\partial^2 l}{\partial\eta^2},\; \sigma^2 = (-l_2)^{-1},\; l_3 = \frac{\partial^3 l}{\partial\eta^3}$$

And for the two-parameter case:

$$\begin{aligned}E(u(\eta)/t) \sim u &+ \frac{1}{2}(u_{11} + 2u_1\rho_1)\sigma_{11} + \frac{1}{2}(u_{12} + 2u_1\rho_2)\sigma_{12} + \frac{1}{2}(u_{21} + 2u_2\rho_1)\sigma_{21} + \frac{1}{2}(u_{22} + 2u_2\rho_2)\sigma_{22} \\ &+ \frac{1}{2}l_{30}\left(u_1\sigma_{11}^2 + u_2\sigma_{11}\sigma_{12}\right) + \frac{1}{2}l_{21}\left(3u_1\sigma_{11}\sigma_{12} + u_2(\sigma_{11}\sigma_{22} + 2\sigma_{12}^2)\right) \\ &+ \frac{1}{2}l_{12}\left(u_1(\sigma_{11}\sigma_{22} + 2\sigma_{12}^2) + 3u_2\sigma_{12}\sigma_{22}\right) + \frac{1}{2}l_{03}\left(u_1\sigma_{12}\sigma_{22} + u_2\sigma_{22}^2\right)\end{aligned} \tag{4-26-b}$$

All the quantities in (4-26-a) and (4-26-b) are to be evaluated at the MLE of the parameter η. All the needed results to get the approximate Bayes estimates are given in the appendix.

4-4-1- The scale parameter

Putting $u(\eta)=u(\theta,\beta)=\theta$ in (4-24) we get the Bayes estimate of the scale parameter. Taking the derivatives of the function u(η) which respect to each parameter in turn yield: $u_1=1$, $u_2=u_{11}=u_{12}=u_{22}=0$. Replacing these derivatives with the above results evaluated at the MLE of the parameters in (4-26-b) gives:

$$\tilde{\tilde{\theta}} \approx \hat{\theta}+\Delta\hat{\theta} \qquad (4\text{-}27\text{-}a)$$

By the same way we put $u(\eta)=u(\theta,\beta)=\theta^2$ (which has derivatives $u_1=2\theta$, $u_2=2$, $u_{11}=u_{12}=u_{22}=0$) to get its posterior second moment.

$$E(\theta^2/t) \approx \hat{\theta}^2+\Delta\hat{\theta}^2 \qquad (4\text{-}27\text{-}b)$$

If we suppose that the shape parameter β is known, the approximate Bayes estimate of the scale parameter and its variance are given by:

$$\tilde{\tilde{\theta}}=\tilde{\theta}=\hat{\theta}=\frac{n(\beta+1)}{\beta\sum_{i=1}^{n} t_i^{\beta}} \qquad (4\text{-}27\text{-}a)$$

$$Var(\hat{\theta})=Var(\tilde{\theta})=Var(\tilde{\tilde{\theta}})=\frac{\theta^2\beta}{n(\beta+1)} \qquad (4\text{-}27\text{-}b)$$

This means that the MLE, the Bayes and approximate Bayes estimates are identical and have the same variance.

4-4-2- The shape parameter

Setting $u(\eta)=u(\theta,\beta)=\beta$ (which has derivatives $u_2=1$, $u_1=u_{11}=u_{12}=u_{22}=0$) in (4-24) gives the Bayes estimate of the shape parameter. Using the derivatives of the function u(η) and the results given in the appendix. Replacing them in (4-26-b) gives:

$$\tilde{\tilde{\beta}} \approx \hat{\beta}+\Delta\hat{\beta} \qquad (4\text{-}28\text{-}a)$$

and we put also $u(\eta)=\beta^2$ in order to obtain the variance, the derivatives of the function "u" in this case are: $u_2=2\beta$, $u_{22}=2$, $u_1=u_{11}=u_{12}=0$. This gives:

$$E(\beta^2/t) \approx \hat{\beta}^2+\Delta\hat{\beta}^2 \qquad (4\text{-}28\text{-}b)$$

If we suppose that the scale parameter θ is known we get:

$$\Delta\hat{\beta} = \frac{1}{2}\left(D + \frac{nA}{\hat{\beta}^3}\right)^{-1}\left(\left(C + \frac{nB}{\hat{\beta}^4}\right)\left(D + \frac{nA}{\hat{\beta}^3}\right)^{-1} - \frac{2}{\hat{\beta}}\right) \quad (4\text{-}28\text{-}c)$$

$$\Delta\hat{\beta}^2 = \left(D + \frac{nA}{\hat{\beta}^3}\right)^{-1}\left(\left(\hat{\beta}C + \frac{nB}{\hat{\beta}^3}\right)\left(D + \frac{nA}{\hat{\beta}^3}\right)^{-1} - 1\right) \quad (4\text{-}28\text{-}d)$$

With:

$$A = -2\ln\theta + 2\Psi(\frac{1}{\hat{\beta}}) + \frac{\Psi'(\frac{1}{\hat{\beta}})}{\hat{\beta}}, \quad B = \frac{\Psi''(\frac{1}{\hat{\beta}})}{\hat{\beta}^2} + 6\frac{\Psi'(\frac{1}{\hat{\beta}})}{\hat{\beta}} - 6\ln\theta, \quad C = \frac{4n}{\hat{\beta}^3} - \theta\sum_{i=1}^{n}(\ln(t_i))^3 t_i^{\hat{\beta}}$$

$$D = \frac{2n}{\hat{\beta}^2} + \theta\sum_{i=1}^{n}(\ln(t_i))^2 t_i^{\hat{\beta}} \quad (4\text{-}28\text{-}e)$$

4-4-3- The reliability function:

To find the estimate of the reliability function we set $u(\eta) = u(\beta,\theta) = R(t)$ in (4-24) (the derivatives of the reliability function are given in the appendix such that $u_1 = R_1$, $u_2 = R_2$, $u_{11} = R_{11}$, $u_{12} = u_{21} = R_{12}$, $u_{22} = R_{22}$) this yield:

$$\tilde{\tilde{R}}(t) \approx \hat{R}(t) + \Delta\hat{R}(t) \quad (4\text{-}29\text{-}a)$$

To get the second posterior moment of the reliability function we put $u(\eta) = u(\beta,\theta) = R^2(t)$ the first and the second derivatives of this function are: $u_1 = 2RR_1$, $u_1 = 2RR_1$, $u_{11} = 2(R_1^2 + R_{11}R)$, $u_{12} = u_{21} = 2(R_1R_2 + R_{12}R)$ and $u_{22} = 2(R_2^2 + R_{22}R)$ we get:

$$E(R^2(t)/t) = \hat{R}^2(t) + \Delta\hat{R}^2(t) \quad (4\text{-}29\text{-}b)$$

- If we suppose that the shape parameter is known we get:

$$\Delta\hat{R}(t) = \frac{t\hat{f}(t)}{2n(\hat{\beta}+1)}\left(\hat{\theta}t^{\hat{\beta}} - \frac{1}{\hat{\beta}}\right) \quad (4\text{-}29\text{-}c)$$

$$\Delta\hat{R}^2(t) = \frac{t\hat{f}(t)}{n(\hat{\beta}+1)}\left(\frac{t\hat{f}(t)}{\hat{\beta}} + \left(\hat{\theta}t^{\hat{\beta}} + \frac{1}{\hat{\beta}}\right)\hat{R}(t)\right) \quad (4\text{-}29\text{-}d)$$

Where $\hat{f}(t)$ is the WLB density evaluated at the MLE of the unknown parameters.

- If the scale parameter is assumed to be known we get

$$\Delta\hat{R}(t) = \frac{1}{2}\left(D + \frac{nA}{\hat{\beta}^3}\right)^{-1}\left(\hat{R}_{22} + \hat{R}_2\left(\left(C + \frac{nB}{\hat{\beta}^4}\right)\left(D + \frac{nA}{\hat{\beta}^3}\right)^{-1} - \frac{2}{\hat{\beta}}\right)\right) \qquad \text{(4-29-e)}$$

$$\Delta\hat{R}^2(t) = \left(D + \frac{nA}{\hat{\beta}^3}\right)^{-1}\left(\hat{R}_2^2 + \hat{R}\left(\hat{R}_{22} + \hat{R}_2\left[\left(C + \frac{nB}{\hat{\beta}^4}\right)\left(D + \frac{nA}{\hat{\beta}^3}\right)^{-1} - \frac{2}{\hat{\beta}}\right]\right)\right) \qquad \text{(4-29-f)}$$

5-NUMERICAL EXAMPLE

To illustrate the above formulas and methods the following data were taken from Gupta and Akman (1998), they represent million of revolutions to failure for 23 ball bearings in fatigue test:

17.880	28.920	33.000	41.520	42.120	45.600	48.480	51.840
51.960	54.120	55.560	67.800	68.640	68.640	68.880	84.120
93.120	98.640	105.120	105.840	127.920	128.040	173.400	

These data have been previously fitted assuming Weibull, lognormal, Inverse Gaussian and length-biased inverse Gaussian. The Kolmogorov-Smirnov test doesn't reject that this data come from a WLB distributon. Some properties of the sample were computed such as the mean $\overline{T} = 72.224$, the variance V(T)=1.344×10^3, the Skewneess $Sk = 1.008$, and kurtosis $Kur = 0.926$; from the values of these two last coefficients the distribution of this data is positive skewed right and leptokurtic. The parameters of the sample were estimated numerically since there was no closed form for them (except when β is known), the system (4-1-a) and (4-1-b) was solved numerically and yields the moment estimates $\beta^* = 1.562$ *and* $\theta^* = 1.847\times10^{-3}$, which were used as initial values for the normal equations in (4-4-a) and (4-4-b) to obtain the maximum likelihood estimates. Bayes and approximate Bayes estimates were also obtained and the results are given in the following table:

Table 1

MLE, Bayes and approximate Bayes estimates of the parameters and their variances**.**

Estimates / Parameters	MLE[1]	BE[2]	ABE[3]
θ	1.768×10^{-3} (6.741×10^{-6})	2.037×10^{-3} (2.652×10^{-6})	3.443×10^{-3} (3.937×10^{-6})
β	1.571 (0.093)	1.471 (8.697×10^{-3})	1.596 (0.093)

* (.): Indicates the variance, $_1$: MLE: maximum likelihood estimates 2:BE: Bayesian estimates, 3: ABE: Approximate Bayes estimate.

We observe that the estimates of the shape and the scale parameters are close and the Bayes one have the smallest variance for the two parameters.

The reliability function was evaluated for certain values of time by both classical and Bayesian methods, the results are given in table 2 below:

Table 2

Moment, MLE, Bayes and approximate Bayes estimates of the reliability function with variances

Estimates / Time t	$R^*(t)$ (moment estimate)	$\hat{R}(t)$ (MLE)	$\tilde{R}(t)$ (Bayes estimate)	$\tilde{\tilde{R}}(t)$ (Approximate Bayes estimate)
10	0.992 -	0.992 (2.487(-4))*	0.976 (1.976 (-3))	0.990 (3.246(-5))
15	0.979 -	0.979 (1.834(-3))	0.949 (8.534(-3))	0.974 (1.812(-4))
20	0.951 -	0.958 (7.171(-3))	0.914 0.018	0.952 (5.336(-4))
50	0.694 -	0.695 (0.253)	0.649 (0.082))	0.693 (6.224(-3))
80	0.370 -	0.370 (0.502)	0.405 (0.089)	0.380 (6.464(-3))
100	0.210 -	0.209 (0.384)	0.283 (0.072)	0.223 (4.443(-3))
173	0.012 -	0.011 (8.534(-3))	0.065 (0.016)	0.019 (1.185(-4))

*2.487(-4) = 2.487×10^{-4}.

From table (2) we observe that $R^*(t), \hat{R}(t)$ and $\tilde{\tilde{R}}(t)$ are indistinguishable, while $\tilde{R}(t)$ presents a slight difference. The approximate method gives the smallest variances for all values of "t".

APPENDIX

The following results are useful to compute the approximate Bayes estimates

A-1- The derivatives of the log likelihood function:

The log likelihood function of the WLB distribution was given in equation (4-3) in section four. The second derivatives of this function evaluated at the MLE are given by:

$$l_{20} = \frac{\partial^2 l}{\partial \theta^2} = -\frac{n(\frac{1}{\beta}+1)}{\theta^2} \quad , \quad l_{11} = \frac{\partial^2 l}{\partial \theta \partial \beta} = -\sum_{i=1}^{n} \ln(t_i)\, t_i^{\beta} - \frac{n}{\theta \beta^2}$$

$$l_{02} = \frac{\partial^2 l}{\partial \beta^2} = -\frac{2\,n}{\beta^2} - \theta \sum_{i=1}^{n} (\ln(t_i))^2 \,(t_i^{\beta}) - \frac{n}{\beta^3}\left(-2\ln(\theta) + 2\Psi(\frac{1}{\beta}) + \frac{\Psi'(\frac{1}{\beta})}{\beta} \right) \tag{A-1}$$

Taking the negatives of these quantities will give the observed information matrix which will be inverted to find the observed variance-covariance matrix with elements:

$$V = \begin{bmatrix} -\hat{l}_{20} & -\hat{l}_{11} \\ -\hat{l}_{11} & -\hat{l}_{02} \end{bmatrix}^{-1} = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} \tag{A-2}$$

The third derivatives of the log likelihood function evaluated at the MLE's are given by:

$$l_{30} = \frac{\partial^3 l}{\partial \theta^3} = 2n \frac{(\frac{1}{\beta}+1)}{\theta^3} \quad , \quad l_{12} = \frac{\partial^3 l}{\partial \theta \partial \beta^2} = \frac{2n}{\theta \beta^3} - \sum_{i=1}^{n} \left(\ln t_i\right)^2 t_i^{\beta} \, , \qquad l_{21} = \frac{\partial^3 l}{\partial \theta^2 \partial \beta} = \frac{n}{\beta^2 \theta^2}$$

$$l_{03} = \frac{\partial^3 l}{\partial \beta^3} = \frac{4n}{\beta^3} - \theta \sum_{i=1}^{n} (\ln(t_i))^3 \, t_i^{\beta} + \frac{n}{\beta^4}\left(\frac{\Psi''(\frac{1}{\beta})}{\beta^2} + \frac{6\Psi'(\frac{1}{\beta})}{\beta} + 6\Psi(\frac{1}{\beta}) - 6\ln\theta \right) \tag{A-3}$$

A-2- The derivatives of the logarithm of the prior function:

The logarithm of the prior density is given by:

$$\rho(\theta,\beta) = \ln(\pi(\theta,\beta)) = -\ln\theta - \ln\beta$$

Differentiating this function with respect to each parameter in turn we get:

$$\rho_1 = \frac{\partial\rho(\theta,\beta)}{\partial\theta} = -\frac{1}{\theta} \quad , \qquad \rho_2 = \frac{\partial\rho(\theta,\beta)}{\partial\beta} = -\frac{1}{\beta} \tag{A-4}$$

A-3- The derivatives of the reliability function:

By differentiating the reliability function given in (1-5) we get with respect to θ and β in turn we get:

A-3-1- The first derivatives:

$$R_1 = \frac{\partial R}{\partial\theta} = -\frac{\beta\, t^{\beta+1}\, \theta^{\frac{1}{\beta}} \exp(-\theta t^{\beta})}{\Gamma(\frac{1}{\beta})} = -\frac{t\, f(t)}{\beta\theta}$$

$$R_2 = \frac{\partial R}{\partial\beta} = -\left(\frac{t\ln(t)}{\beta}\right) f(t) + \frac{I_1}{\beta\,\Gamma(\frac{1}{\beta})} - \left(\frac{\Psi(\frac{1}{\beta})}{\beta^2} + \frac{1}{\beta}\right)(1-R) \tag{A-5}$$

A-3-1-The second derivatives:

$$R_{11} = \frac{\partial^2 R}{\partial\theta^2} = -\frac{t\, f(t)}{\beta\theta}\left(-t^{\beta} + \frac{1}{\beta\theta}\right)$$

$$R_{12} = \frac{\partial^2 R}{\partial\theta\partial\beta} = -\frac{t\, f(t)}{\beta\theta}\left(\ln(t)\left(1-\theta t^{\beta}\right) - \frac{\ln\theta}{\beta^2} + \frac{\Psi(\frac{1}{\beta})}{\beta^2} + \frac{1}{\beta}\right)$$

$$R_{22} = \frac{\partial^2 R}{\partial\beta^2} = -\frac{t\ln t\, f(t)}{\beta}\left(\ln t\left(1-\frac{1}{\beta}-\theta t^{\beta}\right) + \frac{2}{\beta^2}\left(\Psi(\frac{1}{\beta}) - \ln\theta + \beta\right)\right) + \frac{\left(2 I_1 \Psi(\frac{1}{\beta}) - I_2\right)}{\beta^3\,\Gamma(\frac{1}{\beta})}$$

$$+\frac{(1-R)}{\beta^2}\left(\frac{\Psi'(\frac{1}{\beta})}{\beta^2} - \left(\frac{\Psi(\frac{1}{\beta})}{\beta}\right)^2\right) \tag{A-6}$$

Where:

$$I_1 = \int_0^{\theta t^{\beta}} \ln(x)\, x^{\frac{1}{\beta}} e^{-x} dx \ , \qquad I_2 = \int_0^{\theta t^{\beta}} (\ln(x))^2\, x^{\frac{1}{\beta}} e^{-x} dx$$

REFERNCES

Abramowitz, M. and Stegun, I.R. (1970). *Handbook of Mathematical functions, with formulas, graphs, and mathematical tables*. Dover Publications, Inc.,New york.

Frank, H. and Althoen, S.C., (1994). *Statistics concepts and applications*. Cambridge University Press, Great Britain.

Glaser, R.E. (1980). Bathtub and related failure rate characterization. ***J. Amer. Statistical Assoc***, Vol.75, 667-672.

Gupta, R.C. and Akman, H.O. (1995). On the reliability studies of a weighted inverse Gaussian model. *Journal of Statistical Planning and Inference* 48, 69-83.

Gupta, R.C. and Kirmani, S. N. V. A. (1990). The role of weighted distribution in stochastic modeling. *Commun. Statist. –Theory Meth.,* 19(9), 3147-3162.

Gupta, R. C. and Tripathi, R.C. (1990). Effect of length-biased sampling on the modeling error. *Communication in statistics –Theory Meth*., 19(4), 1483-1491.

Gupta, R. C. and Tripathi, R.C. (1996). Weighted bivariate logarithmic series distributions. *Commun. Statist. –Theory Meth.,* 25(5), 1099-1117.

Gradshteyn, I.S. and Ryzhik, I.M. (1965): Table of integrals, series, and products. Academic press Inc., New York and London.

Jain, K., Singh,H. and Bagai, I. (1989). Relations for reliability measures of weighted distributions. . *Commun. Statist. –Theory Meth.*, 18(2), 4393-4412.

Kapur, J.N. and Saxena, H.C. (2001). *Mathematical statistics*, 20th edition, S. Chand & Company LTD New Delhi, India, reprint (2003).

Khatree, R. (1989). Characterization of Inverse-Gaussian and Gamma distributions Through their length-biased distributions. *IEEE Transactions on Reliability*, 38(5), 610-611.

Lindley, D.V., (1980).Approximate Bayesian Methods. *Trabojos de Estadistica,* Vol.31 223-245.

Olyede, B.O. and George, E.O.(2002). On stochastic Inequalities and comparisons of reliability measures for weighted distributions. *Mathematical Problems in Engineering*. Vol.8, 1-13.

Patill, G.P. (2002). Weighted distributions. *Encyclopedia of Environmetics*, Vol.4, 2369-2377 Johon Wiley & Sons.

Patill, G.P. and Rao,C.R. (1978). Weighted distributions and size-biased sampling with application to wildlife populations and human families. *Biometrics*,34,179- 189.

Patill, G.P. and Rao,C.R. and Ratnaparkhi, M.V. (1986). On discrete weighted distributions and their use in model choice for observed data. . *Commun. Statist. –Theory Meth*., 15(3), 907-918.

Stacy, E.W. (1962): A generalization of the gamma distribution, *Ann. Math. Stat., Vol.*33, 1178-1192.