On the Optimal Design of Step-Stress Partially Accelerated Life Tests for the Gompertz Distribution with Type-I Censoring

Ali A. Ismail*

Abstract

This paper studies simple time-step stress Partially Accelerated Life Tests (PALT). It is assumed that the lifetimes of test units follow a two-parameter Gompertz distribution and are type-I censored. Maximum Likelihood Estimates of model parameters are obtained. Estimates of the variances of the estimators are also presented. In addition, optimum test plans for simple time-step stress test are developed. Finally, for illustration, numerical examples are provided.

Key Words and Phrases: Reliability; Gompertz distribution; partial acceleration; step-stress test; maximum likelihood estimation; Fisher information matrix; generalized asymptotic variance; Newton-Raphson method; optimum test plan; type-I censoring; numerical illustration.

1. Introduction

For products having a high reliability, the test of product life under normal use often requires a long period of time. The PALT consists of a variety of test methods for shortening the life of products or hastening the degradation of their performance. The aim of such testing is to quickly obtain data, properly modeled and analyzed, which yield desired information on product life or performance under normal use. PALT can be carried out using constant-stress, step-stress, or progressive-stress (linearly increasing stress). According to Nelson (1990), the stress can be applied in various ways. One way to accelerate failure is step-stress, which increases the stress applied to test product in a specified discrete sequence. Generally, as indicated by Xiong and Ji (2004), a test unit starts at a specified low stress. If the unit does not fail at a specified time, stress on it is raised and held a specified time. Stress is repeatedly increased and held, until the test unit fails or a censoring time is reached.

This paper considers simple time-step stress PALT that uses only two stress levels. Under step-stress PALT a test unit is first run at normal use condition and, if it does not fail for a specified time τ, then it is run at accelerated use condition until failure occurs or the observation is censored. The objective of such experiment is to collect more failure data in a limited time without necessarily using a high stress to all test units.

*Assistant Professor of Statistics, Department of Statistics, Faculty of Economics & Political Science, Cairo University, Giza, Egypt. [e-mail: aismail**100**@yahoo.com].

There is an amount of literature on such partially accelerated life tests. Goel (1971) considered the estimation problem of the acceleration factor using both maximum likelihood and Basyian methods for items having exponential distribution and uniform distribution. The estimates of the parameters of the lifetime distributions were obtained in the case of step-stress PALT under complete sampling. Also, the optimal PALT plan was considered. DeGroot and Goel (1979) used the Bayesian approach, with different loss functions, to estimate the parameters of the exponential distribution and the acceleration factor for step-stress PALT in the case of complete sampling.

Also, PALT was studied with type-I censored data by few authors. Bai and Chung (1992) used the maximum likelihood method to estimate the scale parameter and the acceleration factor for exponentially distributed lifetimes in the case of step and constant PALTs under type-I censoring. They also considered the problem of optimally designing the two types of PALT that terminates at a predetermined time. Bai, Chung and Chun (1993) considered the estimation problem of parameters for items having lognormally distributed lives. The parameters of this lifetime distribution and the acceleration factor were estimated by the method of maximum likelihood in step PALT under Type-I censoring. They determined the stress change-time that minimizes the Generalized Asymptotic Variance (GAV) of MLEs of the model parameters. Aly et al. (1996) considered the maximum likelihood method for estimating the acceleration factor and the parameters of Weibull distribution in step PALT. They obtained the maximum likelihood estimates in the case of type-I censoring.

Abdel-Ghaly et al. (1997) used the Bayesian approach for estimating the Weibull distribution parameters given that the shape parameter is known. They studied the estimation problem in step-stress PALT under both type-I and type-II censored data. Madi (1997) applied the Gibbs sampling approach to the partially accelerated life testing (PALT). This sampling approach was proposed as a general method for Bayesian calculations. He derived empirical Bayes estimators for the failure of the exponential lifetime distribution under normal conditions. Abdel-Ghani (1998) considered the estimation problem of the parameters of Weibull distribution and the acceleration factor for both step-stress PALT and constant-stress PALT using maximum likelihood method and Bayesian approach under type-I and type-II censored data.

Recently, Abdel-Ghaly et al. (2002) studied both the estimation and optimal design problems for the Pareto distribution under step-stress PALT with type-I censoring. They derived the maximum likelihood estimates of the model parameters. Also, they obtained the optimal stress change-time that minimizes the GAV of the MLEs of the model parameters. In addition, Abdel-Ghaly et al. (2003) obtained the maximum likelihood estimates and developed optimum test plans for simple time step-stress PALT with type-II censoring for items having Pareto distribution as a lifetime model. Such plans minimize the GAV of the MLEs of the model parameters. Tahir (2003) considered a two-stage sequential procedure for estimating the failure rate in the step-stress PALT model under the squared error loss. In the first stage, the units are put under normal stress until time t, where t is determined as a stopping time which minimizes the expected loss plus cost of running the test. In the second stage, the stress is raised to a higher level for those units that did not fail by time t and is held constant until they all fail. The accumulated data are then used to estimate the failure rate with the Bayes estimator. It is assumed that the lifetimes of the test units are independent and exponentially distributed random variables with a common failure rate. In addition, the failure rate is also considered as a random variable having a gamma distribution.

Abdel-Ghani (2004) considered the estimation problem of the log-logistic distribution parameters under step PALT. The parameters of this distribution and the acceleration factor were estimated using the maximum likelihood method in the case of type-I censored data. More recently, Ismail (2004) used both maximum-likelihood approach and Bayesian approach for estimating the acceleration factor and the parameters of Pareto distribution of the second kind. This work was conducted under both step-stress PALT and constant-stress PALT in the case of type-I and type-II censored data. Also, he considered the problem of optimal designs for the two types of PALT.

This paper concentrates on both the estimation problem and the optimal design problem in the case of Gompertz distribution under step-stress PALT using type-I censored data. The paper can be organized as follows: In Section 2 the Gompertz distribution is introduced as a lifetime model and the test method is also described. The maximum likelihood estimates of the model parameters are obtained in Section 3. In Section 4 optimum test plans for simple time-step stress PALT are developed. Numerical examples are provided in Section 5 to illustrate the theoretical results. Finally, Section 6 is devoted to the concluding remarks.

2. *The Model and Test Method*

Notations

n	total number of test items in a PALT
η	censoring time of a PALT
T	lifetime of an item at normal use condition
Y	total lifetime of an item in a step PALT
$f(t)$	probability density function at time t at normal use condition
$R(t)$	reliability function at time t at normal use condition
$h(t)$	hazard (failure) rate at time t at normal use condition
β	acceleration factor ($\beta > 1$)
τ	stress change-time in a step PALT ($\tau < \eta$)
P_u	Probability that an item fails at normal use condition
P_a	Probability that an item fails at accelerated use condition
GAV	generalized asymptotic variance
MLE	maximum likelihood estimates/estimators
$\wedge$	implies a maximum likelihood estimate
$\downarrow$ (.)	evaluated at (.)
θ, α	the parameters of Gompertz distribution ($\theta > 0$ and $\alpha > 0$)
y_i	observed value of the total lifetime Y_i of item i , $i = 1, \ldots, n$
n_u, n_a	numbers of items failed at normal use and accelerated use conditions, respectively

$y_{(1)} \le \ldots \le y_{(n_u)} \le \tau \le y_{(n_u+1)} \le \ldots \le y_{(n_u+n_a)} \le \eta$ ordered failure times

2.1 *The Gompertz Distribution: As a Lifetime model*

The Gompertz distribution plays an important role in modeling survival times, human mortality and actuarial tables. According to the literature, the Gompertz distribution was formulated by Gompertz (1825) to fit mortality tables. Recently, many authors have contributed to the statistical methodology and characterization of this distribution. For example, Read (1983), Gordon (1990), Makany (1991), Rao and Damaraju (1992), Franses (1994) and Wu and Lee (1999). Garg et al. (1970) studied the properties of the Gompertz distribution and obtained the maximum likelihood estimates for the parameters. Chen (1997) developed an exact confidence interval and an exact joint confidence region for the parameters of the Gompertz distribution under type-II censoring.

In this paper the lifetimes of the test items are assumed to follow a Gompertz distribution with probability density function (pdf) as follows:

$$f(t;\theta,\alpha) = \theta\, e^{\alpha t} \exp\{-(\theta/\alpha)(e^{\alpha t}-1)\}, \qquad t>0,\ \theta>0,\ \alpha>0, \tag{1}$$

This distribution does not seem to have received enough attention, possibly because of its complicated form [Garg et al. (1970)]. It is worth noting that when $\alpha \rightarrow 0$, the Gompertz distribution will tend to an exponential distribution [Wu et al. (2003)]. The two-parameter Gompertz model is a commonly used survival time distribution in actuarial science and reliability and life testing [Ananda et al. (1996)]. There are several forms for the Gompertz distribution given in the literature. Some of these are given in Johnson et al. (1994). The pdf formula given in equation (1) is the commonly used form and it is unimodal. It has positive skewness and an increasing hazard rate function. In addition, the Gompertz distribution can be interpreted as a truncated extreme value type-I distribution [Johnson et al. (1994)]. According to Jaheen (2003), The Gompertz distribution has been used as a growth model, especially in epidemiological and biomedical studies.

The Gompertz distribution is a theoretical distribution of survival times. Gompertz (1825) proposed a probability model for human mortality, based on the assumption that the "average exhaustion of a man's power to avoid death to be such that at the end of equal infinetely small intervals of time he lost equal portions of his remaining power to oppose destruction which he had at the commencement of these intervals" [Johnson et al. (1995)]. Also, according to Walker and Adham (2001), the Gompertz distribution has many applications, particularly in medical and actuarial studies. However, there has been little recent work on the Gompertz in comparison with its early investigation. Osman (1987) derived a compound Gompertz model by assuming that one of the parameters of the Gompertz distribution is a random variable following the gamma distribution. He studied the properties of compound Gompertz distribution and suggested its use for modelling lifetime data and analyzing the survivals in heterogeneous populations.

The reliability function of the Gompertz distribution takes the form:

$$R(t) = \exp\{-(\theta/\alpha)(e^{\alpha t}-1)\}\quad , \tag{2}$$

and the corresponding hazard rate is given by:

$$h(t) = \theta\, e^{\alpha t}\quad ; \tag{3}$$

Thus, the hazard rate increases exponentially over time.

2.2 ***The Test Method***

In the case of step-stress PALT, the test procedure and its assumptions are described as follows:

- ***Test procedure***

1. Each of the n test items is first run at normal use condition.
2. If it does not fail at normal use condition by a pre-specified time τ then it is put on accelerated use condition and run until the test is terminated.

In this case, if the item has not failed by some pre-specified time τ (τ is called stress change-time), the test condition is switched to a higher level of stress and it is continued until failure occurs or the observation is censored. The effect of this switch is to multiply the remaining lifetime of the item by the inverse of an acceleration factor β, which is the ratio of the hazard rate at accelerated condition to that at normal use condition ($\beta > 1$). Thus, the total lifetime of a test item, denoted by Y, passes through two stages, the first stage is the normal use condition and the second one is the accelerated use condition, respectively.

- ***Assumptions***

1. The total lifetime Y of an item is as follows :

$$Y = \begin{cases} T & \text{if } T \leq \tau \\ \tau + \beta^{-1}(T - \tau) & \text{if } T > \tau, \end{cases} \tag{4}$$

where T is the lifetime of an item at normal use condition. This model is called the *tampered random variable* (TRV) model. It was proposed by DeGroot & Goel (1979).

2. The lifetimes $Y_1, \ldots, Y_n$ of the n test items are independent and identically distributed random variables (i.i.d. r.v.'s).

3. Maximum Likelihood Estimation of the Model Parameters

As indicated by Grimshaw (1993), the ML method is commonly used for most theoretical models and kinds of censored data. Although the exact sampling distribution of maximum likelihood estimators (MLE) is sometimes unknown, MLE have the desirable properties of being consistent and asymptotically normal for large samples under appropriate regularity conditions.

The lifetime of test unit is assumed to follow the two-parameter Gompertz distribution with pdf given in equation (1) [see Garg et al. (1970)] . Therefore, the probability density function of total lifetime Y of an item in a step-stress PALT is given by:

$$f(y) = \begin{cases} f_1(y) & \text{if } 0 < y \le \tau \\ f_2(y) & \text{if } y > \tau \end{cases} \tag{5}$$

where

$f_1(y) = \theta \exp\{\alpha y - (\theta / \alpha)[\exp(\alpha y) - 1]\}$, which is equivalent form to equation (1),

$f_2(y) = \beta\theta \exp\{\alpha[\beta(y - \tau) + \tau] - (\theta / \alpha)[\exp(\alpha[\beta(y - \tau) + \tau]) - 1]\}$,

that is obtained by the transformation-variable technique using equations (1) and (4),

and

$\beta > 1$, $\theta > 0$ and $\alpha > 0$.

The observed values of the total lifetime Y are given by:

$y_{(1)} \le \ldots \le y_{(n_u)} \le \tau \le y_{(n_u+1)} \le \ldots \le y_{(n_u+n_a)} \le \eta$

Since the total lifetimes $Y_1, \ldots, Y_n$ of n items are i.i.d. r.v.'s, then the general form of the total likelihood function for them can be written as:

$$L(.) \propto \prod_{i=1}^{n_u} f_1(y_i) \times \prod_{i=1}^{n_a} f_2(y_i) \times \prod_{i=1}^{n_c} R(\eta) \quad ,$$

where

$n_c = n - n_u - n_a$.

Therefore, the likelihood function of the sample is given by

$$\begin{aligned} L(\beta,\theta,\alpha) \propto & \prod_{i=1}^{n_u} \theta \exp\{\alpha y_i - (\theta/\alpha)[\exp(\alpha y_i) - 1]\} \\ & \times \prod_{i=1}^{n_a} \beta\theta \exp\{\alpha[\beta(y_i - \tau) + \tau] - (\theta/\alpha)[\exp(\alpha[\beta(y_i - \tau) + \tau]) - 1]\} \\ & \times \prod_{i=1}^{n_c} \exp\{-(\theta/\alpha)(\exp(\alpha\eta) - 1)\} \end{aligned} \tag{6}$$

It is usually easier to maximize the natural logarithm of the likelihood function rather than the likelihood function itself. So, the natural logarithm of the likelihood function can be written as:

$$\ln L = (n_u + n_a)\ln\theta + n_a \ln\beta + \alpha\{\sum_{i=1}^{n_u} y_i + \sum_{i=1}^{n_a}[\beta(y_i - \tau) + \tau]\}$$
$$-(\theta/\alpha)\{\sum_{i=1}^{n_u}[\exp(\alpha y_i) - 1] + \sum_{i=1}^{n_a}[\exp(\alpha[\beta(y_i - \tau) + \tau]) - 1]$$
$$+ n_c[\exp(\alpha[\beta(\eta - \tau) + \tau]) - 1]\} \quad (7)$$

The first derivatives of the natural logarithm of the total likelihood function in (7) with respect to β, θ and α are given by:

$$\frac{\partial \ln L}{\partial \beta} = \frac{n_a}{\beta} + \alpha \sum_{i=1}^{n_a}(y_i - \tau) - \theta\,\psi_1 \;, \quad (8)$$

where

$$\psi_1 = \sum_{i=1}^{n_a}\{(y_i - \tau)\exp(\alpha[\beta(y_i - \tau) + \tau])\}$$

$$+ n_c(\eta - \tau)\exp(\alpha[\beta(\eta - \tau) + \tau]) \;.$$

$$\frac{\partial \ln L}{\partial \theta} = \frac{n_u + n_a}{\theta} - \frac{\psi_2}{\alpha} \;, \quad (9)$$

where

$$\psi_2 = \sum_{i=1}^{n_u}[\exp(\alpha y_i) - 1] + \sum_{i=1}^{n_a}[\exp(\alpha[\beta(y_i - \tau) + \tau]) - 1]$$

$$+ n_c[\exp(\alpha[\beta(\eta - \tau) + \tau]) - 1] \;.$$

$$\frac{\partial \ln L}{\partial \alpha} = \sum_{i=1}^{n_u} y_i + \sum_{i=1}^{n_a}[\beta(y_i - \tau) + \tau] - (\theta/\alpha^2)[\alpha\psi_3 - \psi_2] \;, \quad (10)$$

where

$$\psi_3 = \frac{\partial \psi_2}{\partial \alpha} = \sum_{i=1}^{n_u}[y_i \exp(\alpha y_i)] + \sum_{i=1}^{n_a}\{[\beta(y_i - \tau) + \tau]\exp(\alpha[\beta(y_i - \tau) + \tau])\}$$

$$+ n_c[\beta(\eta - \tau) + \tau][\exp(\alpha[\beta(\eta - \tau) + \tau])] \ .$$

By equating equation (9) to zero, the maximum likelihood estimate of θ can be given by the following estimation equation:

$$\hat{\theta} = \frac{(n_u + n_a)\hat{\alpha}}{\psi_2} \ . \tag{11}$$

By substituting for θ into the two equations (8) and (10) and equating each of them to zero, the system equations are then reduced to the following two non-linear equations:

$$\frac{n_a}{\hat{\beta}} + \hat{\alpha}\sum_{i=1}^{n_a}(y_i - \tau) - \frac{(n_u + n_a)\hat{\alpha}\psi_1}{\psi_2} = 0 \ , \tag{12}$$

and

$$\sum_{i=1}^{n_u} y_i + \sum_{i=1}^{n_a}\hat{\beta}[(y_i - \tau) + \tau] - \frac{(n_u + n_a)(\hat{\alpha}\psi_3 - \psi_2)}{\hat{\alpha}\psi_2} = 0 \ . \tag{13}$$

Obviously, it is very difficult to obtain a closed-form solution for the two non-linear equations (12) and (13). So, iterative procedures must be used to solve these equations, numerically. The Newton-Raphson method is used to obtain the MLE of β and α. Thus, once the values of $\hat{\beta}$ *and* $\hat{\alpha}$ are determined, an estimate of θ is easily obtained from (11).

In relation to the asymptotic variance-covariance matrix of the MLE of the parameters, it can be approximated by numerically inverting the asymptotic Fisher-information matrix *F*. It is composed of the negative second derivatives of the natural logarithm of the likelihood function evaluated at the MLE. Therefore, the asymptotic Fisher-information matrix can be written as follows:

$$F = \begin{bmatrix} -\frac{\partial^2 \ln L}{\partial \beta^2} & -\frac{\partial^2 \ln L}{\partial \beta \partial \theta} & -\frac{\partial^2 \ln L}{\partial \beta \partial \alpha} \\ -\frac{\partial^2 \ln L}{\partial \theta \partial \beta} & -\frac{\partial^2 \ln L}{\partial \theta^2} & -\frac{\partial^2 \ln L}{\partial \theta \partial \alpha} \\ -\frac{\partial^2 \ln L}{\partial \alpha \partial \beta} & -\frac{\partial^2 \ln L}{\partial \alpha \partial \theta} & -\frac{\partial^2 \ln L}{\partial \alpha^2} \end{bmatrix} \downarrow (\hat{\beta}, \hat{\theta}, \hat{\alpha}) \tag{14}$$

The elements of the above matrix F can be expressed by the following equations:

$$\frac{\partial 2 \ln L}{\partial \beta 2} = -\frac{n_a}{\beta 2} - \theta\alpha\{\sum_{i=1}^{n_a}[(y_i - \tau)2 \exp(\alpha[\beta(y_i - \tau) + \tau])]$$

$$+ n_c(\eta - \tau)2 \exp(\alpha[\beta(\eta - \tau) + \tau])\} \ ,$$

$$\frac{\partial 2 \ln L}{\partial \theta 2} = -\frac{n_u + n_a}{\theta 2} \ ,$$

$$\frac{\partial 2 \ln L}{\partial \alpha 2} = -(\theta / \alpha 2)[\alpha\psi_4 - \psi_3] + (\theta / \alpha 4)[\alpha 2\psi_3 - 2\alpha\psi_2] \ ,$$

where

$$\psi_4 = \frac{\partial \psi_3}{\partial \alpha} = \sum_{i=1}^{n_u}[y_i^2 \exp(\alpha\, y_i)] + \sum_{i=1}^{n_a}\{[\beta(y_i - \tau) + \tau]^2 \exp(\alpha[\beta(y_i - \tau) + \tau])\}$$

$$+ n_c[\beta(\eta - \tau) + \tau]^2[\exp(\alpha[\beta(\eta - \tau) + \tau])] \ .$$

$$\frac{\partial 2 \ln L}{\partial \beta \partial \theta} = -\psi_1 \ ,$$

$$\frac{\partial 2 \ln L}{\partial \beta \partial \alpha} = \sum_{i=1}^{n_a}(y_i - \tau) - \theta[\sum_{i=1}^{n_a}\{(y_i - \tau)[\beta(y_i - \tau) + \tau]\exp(\alpha[\beta(y_i - \tau) + \tau])\}$$

$$(\quad + n_c(\eta - \tau)[\beta(\eta - \tau) + \tau]\exp(\alpha[\beta(\eta - \tau) + \tau])] \ ,$$

$$\frac{\partial 2 \ln L}{\partial \theta \partial \alpha} = -(1 / \alpha 2)[\alpha\psi_3 - \psi_2] \ .$$

Therefore, the maximum likelihood estimators of β, θ and α have an asymptotic variance-covariance matrix obtained by inverting the Fisher information matrix defined in equation (14).

4. Optimum Test Plan

Now, for the optimal design stage of the test, a new experiment with test units different from those tested in the stage of parameter estimation is conducted. The current aim is to obtain the optimal value of τ based on the outputs of the stage of parameter estimation that are in the same time considered inputs to the optimal design stage of the test.

It is worth noting that the stress change-time τ is a prespecified time for the stage of parameter estimation. But for the optimal design stage of the test, τ is considered a switching parameter that to be optimally determined according to a certain optimality criterion.

This section considers the problem of optimally designing a simple time-step-stress PALT, which terminates at a pre-specified censoring time. Optimum test plan for products having a two-parameter Gompertz lifetime distribution is developed. The optimum criterion is to find the optimal stress change-time τ^* such that the GAV of the MLE of the model parameters at normal use condition is minimized.

The GAV of the MLE of the model Parameters is the reciprocal of the determinant of *F*. That is,

$$\text{GAV}\,(\hat{\beta}, \hat{\theta}, \hat{\alpha}) = \frac{1}{|F|} \tag{15}$$

The minimization of the GAV over τ solves the following equation:

$$\frac{\partial\, GAV}{\partial\, \tau} = 0 \tag{16}$$

In general, the solution to (16) is not in a closed form and therefore requires a numerical method such as Newton-Raphson. The Newton-Raphson method is applied to obtain the optimal stress-change time τ^* which minimizes the GAV. Accordingly, the corresponding optimal numbers of items failed at normal use condition and accelerated use condition can be calculated, respectively, via the following two formulas:

$$nP_u \equiv n[1 - \exp\{-(\hat{\theta} / \hat{\alpha})(e^{\hat{\alpha}\tau^*} - 1)\}] \ ,$$

and

$$nP_a \equiv n[\exp\{-(\hat{\theta} / \hat{\alpha})(e^{\hat{\alpha}\tau^*} - 1)\}][1 - \exp\{-(\hat{\theta} / \hat{\alpha})(e^{\hat{\alpha}\hat{\beta}(\eta - \tau^*)} - 1)\}] \ . \tag{17}$$

5. A Numerical Illustration

The main objective of this section is to make a numerical investigation for illustrating the theoretical results of both estimation and optimal design problems. Considering the type-I censoring, several data sets are generated from Gompertz distribution for different combinations of the true parameter values of β, θ and α. The true parameter values used in this section are (3, 0.1, 0.3) and (7, 0.2, 0.5). Different samples in size (n = 100, 200, 300, 400, 500, 800 and 1000) are considered using 1000 replications for each sample size. Computer program using the Pascal language is prepared and the Newton-Raphson method is used for obtaining the ML estimates of β, θ and α. Therefore, the derived nonlinear logarithmic likelihood equations in (12) and (13) are solved iteratively. Once the values of $\hat{\beta}$ *and* $\hat{\alpha}$ are determined, an estimate of the parameter θ is easily obtained from equation (11). For different sample sizes and different true values of the parameters, the ML estimates of the model parameters and the estimated asymptotic variances of the ML estimators of the model parameters are reported in Table (1).

Results of numerical illustration studies provide insight into the sampling behavior of the estimators. The numerical results indicate that the ML estimates approximate the true values of the parameters as the sample size n increases. Also, as shown from the numerical results, the asymptotic variances of the estimators decrease as the sample size n is getting to be larger.

Also, optimum test plans are developed numerically. The optimal stress change-time, optimal number of items failed at normal use and accelerated use conditions and optimal GAV of the ML estimators of the model parameters are included in Table (2). The numerical results shown in Table (2) demonstrate that the optimal stress change-time, minimizing the GAV of the ML estimators of the model parameters, usually approach the censoring time at which the life test is terminated. That is, testing only at normal use condition. Also, Table (2) presents the optimal GAV which is numerically obtained with τ^* in place of τ for different sized samples. As indicated from the results, the optimal GAV decreases as the sample size increases.

Table (1)
The MLE and estimated asymptotic variances of the ML estimators of the parameters for different sized samples under type-I censoring in step-stress PALT

$(\beta, \theta, \alpha, \tau, \eta)$		(3, 0.1, 0.3, 1.5, 2)		(7, 0.2, 0.5, 1.5, 2)	
n	Parameter	Estimate	Variance	Estimate	Variance
100	β	7.7571	6.9441	13.6051	39.6023
	θ	0.5013	0.0916	0.6712	0.0752
	α	0.6512	0.2801	0.7435	0.2361
200	β	6.6960	5.7783	10.3822	22.3710
	θ	0.4523	0.0651	0.6147	0.0683
	α	0.6201	0.0977	0.7124	0.1891
300	β	4.3622	3.0220	9.1984	11.1076
	θ	0.4134	0.0426	0.5267	0.0491
	α	0.5032	0.0456	0.6941	0.1546
400	β	3.3026	1.7578	8.4305	8.5504
	θ	0.3621	0.0251	0.4867	0.0372
	α	0.4472	0.0272	0.6468	0.1213
500	β	3.2132	1.3538	7.6236	7.5193
	θ	0.3254	0.0172	0.3694	0.0205
	α	0.4169	0.0229	0.5901	0.0904
800	β	3.1852	0.6652	7.3516	4.1268
	θ	0.1963	0.0132	0.2863	0.0183
	α	0.3774	0.0176	0.5527	0.0614
1000	β	3.1383	0.4863	7.1738	3.2056
	θ	0.1704	0.0108	0.2348	0.0111
	α	0.3441	0.0123	0.5139	0.0334

Table (2)
The results of optimal design of step-stress PALT for different sized samples under type-I censoring

$(\beta, \theta, \alpha, \eta)$	(3, 0.1, 0.3, 2)				(7, 0.2, 0.5, 2)			
n	τ^*	nP_u	nP_a	Optimal GAV	τ^*	nP_u	nP_a	Optimal GAV
100	2	87	0	0.4351	2	95	0	0.6041
200	2	167	0	0.3725	2	187	0	0.4701
300	2	228	0	0.3101	2	269	0	0.3276
400	2	276	0	0.2786	2	345	0	0.2631
500	2	319	0	0.1527	2	378	0	0.1271
800	2	355	0	0.0645	2	519	0	0.0349
1000	2	388	0	0.0428	2	560	0	0.0132

6. Concluding Remarks

This paper deals with the problems of estimation and optimally designing simple time-step stress PALT for the Gompertz distribution under type-I censored data. The maximum likelihood estimates of the model parameters were obtained, numerically. Also, optimum test plans were developed under the assumptions of Gompertz lifetimes of test units and type-I censoring. The minimization of the GAV of the MLE of model parameters was adopted as an optimality criterion. As shown from the numerical results, at different sized samples, τ^* always approach the censoring time η. That is, testing only at normal use condition.

In practice, the optimum test plans are important for improving the level of precision in parameter estimation and thus improving the quality of the inference. So, these optimum plans are more useful and more efficient for estimating the life distribution at design stress. The usefulness of optimal designs lies in the fact that they can serve as benchmarks with which to compare other designs. Finally, the problem of designing *failure-step* stress PALT when the lifetime of an item at normal use condition follows a Weibull distribution is currently under study.

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