Moment Properties of Hidden Semimartingale Models (HSM) with GARCH Errors

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Abstract

Following Hamilton (1989), in this paper, we propose a semimartingale model with conditional mean of the observed process driven by a semimartingale and the volatility driven by a GARCH process. Moment properties of these models including kurtosis are studied in some detail.

Keywords: GARCH; RCA-GARCH ; Hidden Semimartingale Model; Nonlinear time series; Semimartingales; Volatility

1 Introduction

Recently, moment properties of Random Coefficient GARCH (RCA-GARCH) models, have been studied in Thavaneswaran et al. (2005). Following the fascinating paper by Hamilton (1989) who models changes in regimes via Markov switching, we consider the situation where the volatility is also changing. Building upon the ideas developed by Abraham and Thavaneswaran (1991) on state space formulation of non-linear models to draw optimal inference about the unknown parameters (see Granger(1998) for details), we propose the estimating function method based inference for the proposed HSM of volatilities.

2 Hidden Semimartingale Models

Consider the random coefficient autoregressive(RCA) model given in Thavaneswaran and Abraham (1988):

$$y_t - \theta_t f(t, F^y_{t-1}) \quad = \quad e_t \qquad (2.1)$$

where $\{\theta_t\}$ is a more general stochastic sequence and $f(t, F_{t-1}^y)$ is a function of the past. When θ_t is a moving average (MA) sequence of the form

$$\theta_t = \theta + \varepsilon_t + \varepsilon_{t-1} \tag{2.2}$$

where θ_t, e_t are square integrable independent random variables and $\{\varepsilon_t\}$ consists of zero mean square integrable *Gaussian random variables* independent of $\{e_t\}$. In this case $E(y_t|F_{t-1}^y)$ depends on the posterior mean $m_t = E(\varepsilon_t|F_t^y)$ and variance $\gamma_t = E[(\varepsilon_t - m_t)^2|F_t^y)$ of ε_t. Assume $y_0 = 0$, then m_t and y_t satisfy the following Kalman-like recursive algorithms,

$$m_t = \frac{\sigma_t^2 f\left(t, F_{t-1}^y\right)\left[y_t - (\theta + m_{t-1}) f\left(t, F_{t-1}^y\right)\right]}{\sigma_e^2 + f^2\left(t, F_{t-1}^y\right)(\sigma_e^2 + \gamma_{t-1})} \tag{2.3}$$

and

$$\gamma_t = \sigma_\varepsilon^2 - \frac{f^2(t, F_{t-1}^y)\sigma_\varepsilon^4}{\sigma_e^2 + f^2\left(t, F_{t-1}^y\right)\left(\sigma_e^2 + \gamma_{t-1}^2\right)} \tag{2.4}$$

where $\gamma_0 = \sigma_\varepsilon^2$ and $m_0 = 0$. Hence, $E(y_t|F_{t-1}^y) = (\theta + m_{t-1}) f(t, F_{t-1}^y)$ and $E(h_t^2|F_t^y) = \sigma_e^2 + f^2(t, F_{t-1}^y)(\sigma_e^2 + \gamma_{t-1})$, where $h_t = y_t - E(y_t|F_{t-1}^y)$, can be calculated recursively. Then the optimal estimating function turns out to be $g_n^* = \sum_{t=2}^{n} h_t a_{t-1}^*$ where,

$$a_{t-1}^* = \frac{E\left[(dh_t/d\theta)\,|F_{t-1}^y\right]}{E\left[h_t^2|F_{t-1}^y\right]}.$$

Thus, the optimal estimate is given by

$$\widehat{\theta}_n = \frac{\sum_{t=2}^n a_{t-1}^* y_t}{\sum_{t=2}^n a_{t-1}^* f(t, F_{t-1}^y)}$$

where

$$a_{t-1}^* = \frac{f\left(t, F_{t-1}^y\right)(1 + (dm_{t-1}/d\theta))}{\left[\sigma_\varepsilon^2 + f^2\left(t, F_{t-1}^y\right)(\sigma_\varepsilon^2 + \gamma_{t-1})\right]}.$$

Since γ_t is independent of θ, the relation

$$\frac{dm_t}{d\theta} = -\frac{\left[\sigma_\varepsilon^2 f^2(t, F_{t-1}^y)(1 + dm_{t-1}/d\theta)\right]}{\left[\sigma_\varepsilon^2 + f^2\left(t, F_{t-1}^y\right)(\sigma_\varepsilon^2 + \gamma_{t-1})\right]}$$

can be used to calculate this derivative recursively.

As can be seen from (2.3) and (2.4), the optimal estimate $\widehat{\theta}_n$ adopts a weighting scheme based on σ_ε^2 and σ_e^2. The superiority of the optimal estimate over the conditional least squares estimate has been demonstrated in Thavaneswaran and Abraham (1988). This shows that optimal inference for RCA models and HSM models can be studied using the estimating function method. Later we show that a GARCH(p, q) model for y_t could be written as an ARMA(r, q) for y_t^2 and hence one could study the optimal inference by combining estimating functions as in Thompson and Thavaneswaran (1999).

Hamilton (1989) used Markov switching to detect changes between positive and negative growth periods in the economy, using a Hidden Markov Model (HMM) of the form

$$y_t = n_t + \varepsilon_t, \tag{2.5}$$

$t = 1, \dots, T$, where ε_t is a zero mean ARIMA(r,1,0) process and n_t (the trend term) is a random walk with drift that switches between two values α_0 and $\alpha_0 + \alpha_1$. That is,

$$n_t = n_{t-1} + \alpha_0 + \alpha_1 S_t, \tag{2.6}$$

$t = 1, \dots, T$, depending on whether the unobserved state of the system S_t is in state 1 or state 0. Furthermore, transitions between the states of S_t is assumed to be governed by the Markov process with: $p_{ij} = Pr(S_{t+1} = j | S_t = i)$. Let the steady-state probabilities be denoted by $\pi = Pr(S_t = 1)$ and $1 - \pi = Pr(S_t = 0)$ respectively. The following lemma gives the corresponding semimartingale form of the binary process S_t.

Lemma 2.1 Let S_t be defined as in (2.6). Then S_t has the following form:

$$S_{t+1} - \pi = d(S_t - \pi) + V_{t+1}, \tag{2.7}$$

where $d = Corr(S_{t+1}, S_t) = p_{11} - p_{01}$ and V_t is a semimartingale with $E[V_{t+1}|S_t = i] = 0$ and $Var[V_{t+1}|S_t = i] = p_{ii}(1 - p_{ii})$, $i = 0, 1$.

Proof: Let $\underline{\pi}' = (1 - \pi \;\; \pi)'$ and let $\mathbf{P} = (p_{ij}),\;\; i, j = 0, 1$ denote the transition probability matrix. The steady-state probability $\pi = E(S_t) = Pr(S_t = 1)$, found by solving the Chapman-Kolmogorov equations $\underline{\pi}'\mathbf{P} = \underline{\pi}'$ is $\pi = \dfrac{p_{01}}{p_{01} + p_{10}}$.

The variance of S_t is $Var(S_t) = E(S_t^2) - \pi^2 = \pi(1-\pi)$.

The correlation between S_t and S_{t+1} follows by observing that $Cov(S_t, S_{t+1}) = E(S_t S_{t+1}) - \pi^2 = Pr(S_t = 1, S_{t+1} = 1) - \pi^2 = Pr(S_{t+1} = 1|S_t = 1)Pr(S_t = 1) - \pi^2 = p_{11}\pi - \pi^2$.

Let $V_{t+1} = S_{t+1} - \pi - d(S_t - \pi)$ and let $i = 0$. Then, $E(V_{t+1}|S_t = 0) = E(S_{t+1}|S_t = 0) - \pi + d\pi = p_{01} - p_{01} = 0$.

Similarly, $Var(V_{t+1}|S_t = 0) = Var(S_{t+1} - dS_t|S_t = 0) = Var(S_{t+1}|S_t = 0) + d^2Var(S_t|S_t = 0) - 2dCov(S_{t+1}, S_t|S_t = 0) = E(S_{t+1}^2|S_t = 0) - E^2(S_{t+1}|S_t = 0) = p_{01}(1-p_{01}) = (1-p_{00})p_{00}$. Using the same reasoning $E(V_{t+1}|S_t = 1) = 0$ and $Var(V_{t+1}|S_t = 1) = (1 - p_{11})p_{11}$.

It is of interest to note that for the usual continuous space AR(1) process $y_t - \mu = \phi(y_{t-1} - \mu) + a_t$, $Var(a_t|y_{t-1}) = \sigma_a^2$, that is, a constant.

2.1 GARCH Models

Now consider the general class of GARCH(p, q) models for the time series ε_t,

$$\varepsilon_t = \sqrt{h_t} Z_t, \tag{2.8}$$

$$h_t = \omega + \sum_{i=1}^{p} \alpha_i \varepsilon_{t-i}^2 + \sum_{j=1}^{q} \beta_i h_{t-j}, \tag{2.9}$$

where Z_t is a sequence of independent, identically distributed random variables with zero mean, unit variance. Let $u_t = \varepsilon_t^2 - h_t$ be the martingale difference and let σ_u^2 be the variance of u_t. Then (2.8) and (2.9) could be written as:

$$\varepsilon_t^2 - u_t = \omega + \sum_{i=1}^{p} \alpha_i \varepsilon_{t-i}^2 + \sum_{j=1}^{q} \beta_i h_{t-j},$$

$$\phi(B)\varepsilon_t^2 = \omega + \beta(B)u_t. \tag{2.10}$$

where, $\phi(B) = 1 - \sum_{i=1}^{r} \phi_i B^i$, $\phi_i = (\alpha_i + \beta_i)$, $\beta(B) = 1 - \sum_{i=1}^{q} \beta_i B^i$ and $r = \max(p, q)$. We shall make the following stationarity assumptions for y_t^2 which has an ARMA(r, q) representation.

(A.1) all zeroes of the polynomial $\phi(B)$ lie outside of the unit circle.

(A.2) $\sum_{i=0}^{\infty} \psi_i^2 < \infty$ where the $\psi_i' s$ are obtained from the relation $\psi(B)\ \phi(B) = \beta(B)$ with $\psi(B) = 1 + \sum_{i=1}^{\infty} \psi_i B^i$.

The assumptions ensure that the $u_t' s$ are uncorrelated with zero mean and finite variance and that the ε_t^2 process is weakly stationary. In this case, the autocorrelation function of ε_t^2 will be exactly the same as that for a stationary ARMA(r, q) model. The kurtosis of the GARCH process is denoted by $K^{(\varepsilon)}$ when it exists.

2.2 HSMs with GARCH Errors

Suppose the observed time series is obtained from the process $y_t = n_t + \varepsilon_t$, where ε_t is a zero mean GARCH(p, q) process as in (2.8), (2.9), (2.10) and n_t (the trend term) is a semimartingale as in (2.6). This model is of interest when the conditional variance of the series is changing and the trend term switches between two states such as contraction and expansion in the economy.

In order to calculate the variance and kurtosis for a HSM with GARCH errors in terms of the ψ weights, we have the following theorem.

Theorem 2.1 For the HSM process for trend, specified by (2.6), (2.7), (2.8), and (2.9), under the assumptions of stationarity and finite fourth moment, the kurtosis $K^{(\varepsilon)}$ of the process is given by:

(a) $K^{(\varepsilon)} = \dfrac{E(Z_t^4)}{E(Z_t^4) - [E(Z_t^4) - 1] \sum_{j=0}^{\infty} \psi_j^2}$,

(b) The variance of the ε_t^2 process is $\gamma_0^{\varepsilon^2} = \displaystyle\sum_{j=0}^{\infty} \psi_j^2 \sigma_u^2$, where $\sigma_u^2 = \dfrac{\mu_{\varepsilon^2}^2 (K^{(\varepsilon)} - 1)}{\sum_{j=0}^{\infty} \psi_j^2}$ and $\mu_{\varepsilon^2} = E(\varepsilon_t^2) = \dfrac{\omega}{1 - \phi_1 - \phi_2 - \cdots - \phi_r}$,

(c) Let μ_t^n and σ_n^2 denote the mean and variance of n_t conditional on n_0, respectively. Then $\mu_t^n = n_0 + (\alpha_0 + \pi\alpha_1)t$ and $\sigma_n^2 = \alpha_1^2 \pi(1-\pi)[t + 2\displaystyle\sum_{j=1}^{t-1}(t-j)d^j]$, where $d = p_{11} - p_{01}$.

(d) When Z_t is Gaussian, the kurtosis of the residual process ε_t and of the observed process y_t are greater than 3.

Proof: Part (a) of Theorem 2.1 somewhat parallels the proof of Theorem 2.1 in Thavaneswaran et al.(2005) for kurtosis. Part (b) follows from (2.10),
$\mu = E(\varepsilon_t^2) = \dfrac{\omega}{1-\phi_1-\phi_2-\ldots-\phi_r}$, $K^{(\varepsilon)} = \dfrac{E(\varepsilon_t^4)}{(E(\varepsilon_t^2))^2} = \dfrac{E(\varepsilon_t^4)}{\mu^2}$, and $\sigma_u^2 = \dfrac{\mu^2(K^{(\varepsilon)}-1)}{\sum_{j=0}^{\infty}\psi_j^2}$.

The proof of part (c) follows by using the fact that $E(S_t) = \pi$ and $Var(S_t) = \pi(1-\pi)$ in equation (2.6). In addition, since S_k was shown to have a stationary AR(1) representation in (2.7), $Var(\sum_{k=1}^{t} S_k) = \gamma_0(t + 2(t-1)\rho_1 + 2(t-2)\rho_2 + \ldots + 2\rho_{t-1})$, where $\gamma_0 = Var(S_t)$. Finally, using the fact that $n_t - n_0 = \alpha_0 t + \alpha_1 \sum_{k=1}^{t} S_k$, the results for mean and variance follow.

Part (d) follows from the fact that for any conditionally Gaussian GARCH process $\varepsilon_t|F_{t-1}^{\varepsilon}$, $E(\varepsilon_t^4) = EE(\varepsilon_t^4|\varepsilon_{t-1}) = E3[E(\varepsilon_t^2|\varepsilon_{t-1})]^2 = 3E[E(\varepsilon_t^2|\varepsilon_{t-1})]^2 \geq 3(E[E(\varepsilon_t^2|\varepsilon_{t-1})])^2 = 3(E(\varepsilon_t^2)^2)$.

In analogy with the RCA example, using the combination theorem given in Thompson and Thavaneswaran (1999), (or the combined estimating function given in Thavaneswaran and Thompson (1988)), we can make inference of the model parameters.

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