A New Distribution Form for SURE Estimates

Ahmed H. Youssef
Associate Professor, Department of Applied Statistics and
Econometrics, Institute of Statistical Studies and Research, Cairo University.
ahyoussef@hotmail.com

A general distribution function for the coefficients of seemingly unrelated regression equations (SURE) model when we unrestricted regression (SUUR) equations was studies in this paper. The exact probability density function of the estimators was derived and studied by Zellner (1962, 1972) and it is very hard to study the properties of this distribution. A new form for this probability density function is presented and generalized. The cumulative distribution function, and the r $^{\underline{th}}$ moment, the characteristics function for SUUR equations estimators are derived in a simple form.

Key Words: Seemingly Unrelated Regression Equations model, Seemingly Unrelated Unrestricted Regression Equations, Distribution function.

1. Introduction

To make the discussion of the seemingly unrelated regression equations (SURE) model more specific, we shall consider a set of individual linear multiple regression equations, each explaining some economic phenomenon. This set of regression equations is said to be a simultaneous equations model if one or more of the regressors (explanatory variables) in one or more of the equations is itself the dependent (endogenous) variable associated with another equation in the full system. On the other hand, suppose that none of the variables in the system are simultaneously both explanatory and dependent in nature, and then there may still be interactions between the individual equations if the random disturbances associated with at least some of the different equations are correlated with each other. This possibility was discussed by Zellner (1962), who coined the expression "seemingly unrelated regression equations" to reflect the fact that the individual equations are in fact related to one another, even though superficially they may not seem to be.

Beckwith (1972) and Wildt (1974) used the SURE model to examine the sales response of competing brands to advertising expenditures. Albon and Valentine (1977) studied demand for bank loans in Australia. Hirschberg (1992) used the bootstrap technique to compute the demand equations, while Rilstone and Veall (1996) used bootstrap for improved inference for SURE model. Youssef (1995, 1997) studied the properties of SURE estimators. Brown and Kadiyala (1985) suggested applying the SURE model to event studies in finance and economics. Other surveys of the seemingly unrelated regression model can be found in Judge et al. (1985) and Srivastava and Dwivedi (1979). An excellent and extensive survey of the theoretical results associated with the SURE model is the Srivastava and Giles (1987).

For simplicity, *Zellner*'s considered a two equations model in which the disturbances are normally distributed, to obtain the probability density function for the standardized estimator $\tilde{\beta}_{(i)}$, denoted by *W*. The sampling distribution of the *SUUR* estimator is

$$f(w) = \frac{1}{\sqrt{2\Pi}} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} \sum_{t=0}^{\infty} \frac{(-1)^t}{t!} \frac{\Gamma\left(t+\frac{n+1}{2}\right)}{\Gamma\left(t+\frac{n+2}{2}\right)} \left[\frac{w^2}{2}\right]^t, \quad -\infty \le w \le +\infty . \tag{1}$$

Zellner (1962) obtained the mean and variance for *SUUR* estimator and *Zellner*(1972) corrected the variance that he obtained in 1962.

2. General Exact Results with New Representation

We introduce a general form of the probability density function of the coefficient estimators for seemingly unrelated unrestricted residual (*SUUR*) by using $n = \alpha$, and $z = w / \theta$. So, we have

$$f\left(z,\alpha,\theta\right) = \frac{\theta}{\sqrt{2\pi}} \frac{\Gamma\left(\frac{\alpha+1}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)} \sum_{t=0}^{\infty} \frac{(-1)^t}{t!} \frac{\Gamma\left(t+\frac{\alpha+1}{2}\right)}{\Gamma\left(t+\frac{\alpha+2}{2}\right)} \left[\frac{(\theta z)^2}{2}\right]^t, \quad -\infty \le z \le +\infty, \alpha \succ 0, \theta \succ 0 . \tag{2}$$

It is hard to prove that this is probability density function, so we introduce a new form of the probability density function by using the confluent hypergeometric function, and then we can find the rth moment, and the characteristic function of the probability density function as given in(2). The probability density function of *z*, as given in (2), can be written as:

$$f\left(z,\alpha,\theta\right) = \frac{\theta}{\sqrt{2\pi}} \frac{\left(\Gamma\frac{\alpha+1}{2}\right)^2}{\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(\frac{\alpha+2}{2}\right)} \sum_{t=0}^{\infty} \frac{1}{t!} \frac{\Gamma\left(t+\frac{\alpha+1}{2}\right)\Gamma\left(\frac{\alpha+2}{2}\right)}{\Gamma\left(t+\frac{\alpha+2}{2}\right)\Gamma\left(\frac{\alpha+1}{2}\right)} \left[-\frac{(\theta z)^2}{2}\right]^t$$

$$= c \;.\; {}_1F_1\left(\frac{\alpha+1}{2};\frac{\alpha+2}{2};-\frac{(\theta z)^2}{2}\right), \tag{3}$$

where

$$ {}_1F_1\left(a;b;z\right)=\sum_{r=0}^{\infty}\frac{\left(a\right)_r}{\left(b\right)_r}\frac{z^r}{r!}, $$

$$ \left(a\right)_r=\frac{\Gamma\left(a+r\right)}{\Gamma\left(a\right)}, $$

and

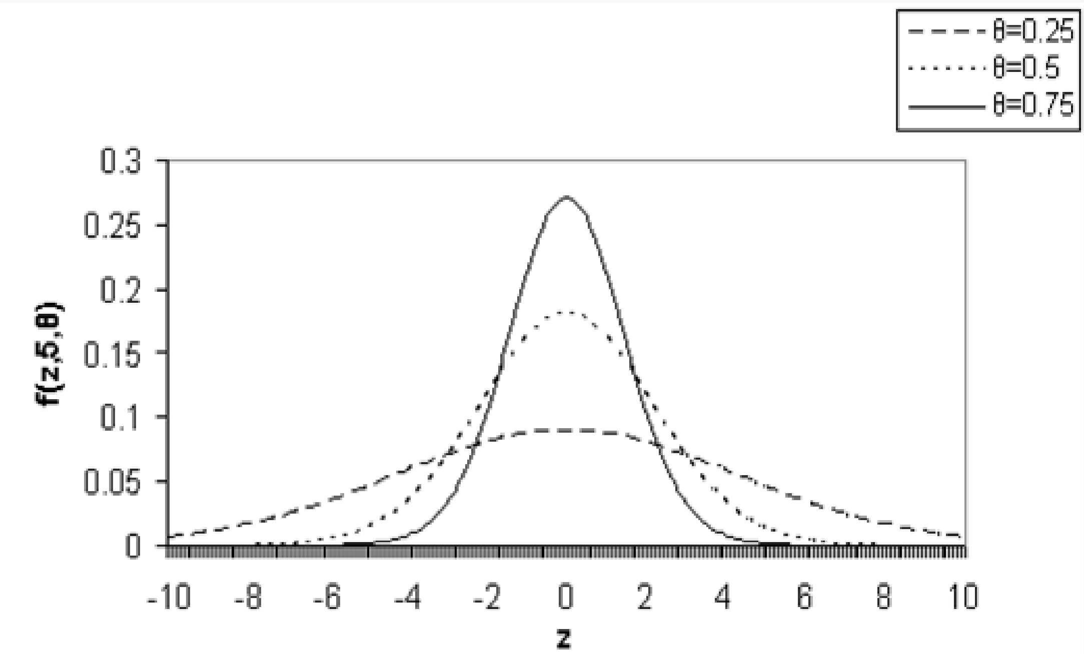
$$ C=\frac{\theta}{\sqrt{2\pi}}\frac{\left(\Gamma\frac{\alpha+1}{2}\right)^2}{\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(\frac{\alpha+2}{2}\right)}. $$

From the properties of the confluent hypergeometric function, we can rewrite (3) as follows:

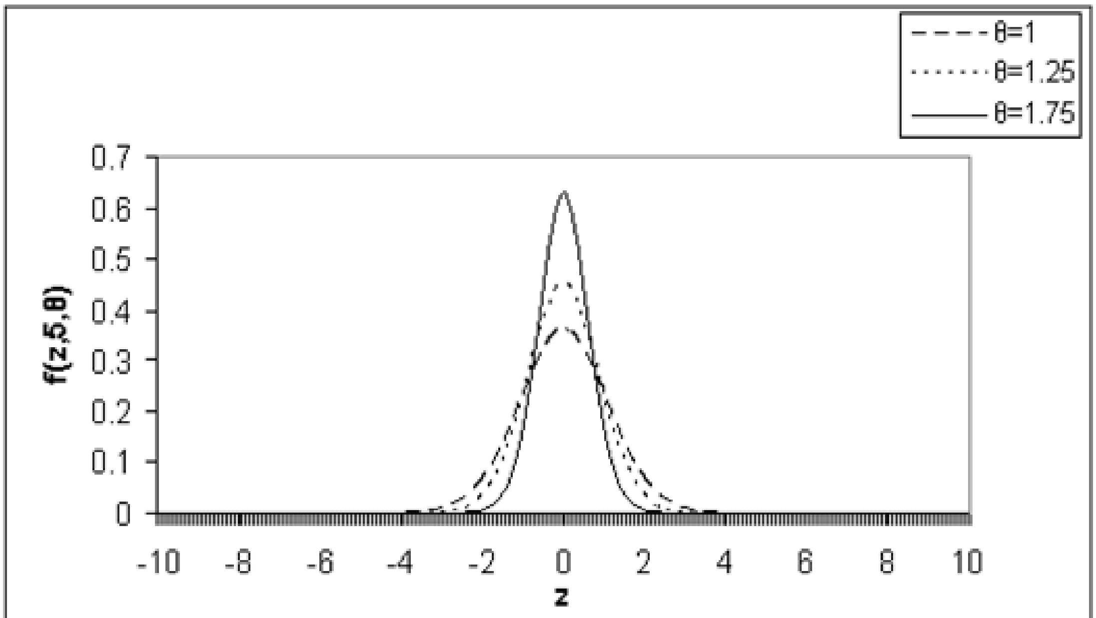
$$ f\left(Z,\alpha,\theta\right)=\frac{\theta\sqrt{2\pi}}{\alpha.\beta^2\left(\frac{1}{2},\frac{\alpha}{2}\right)}\; e^{-\frac{(\theta Z)^2}{2}}\; {}_1F_1\left(\frac{1}{2};\frac{\alpha+2}{2};\frac{\left(\theta Z\right)^2}{2}\right). $$

$$ -\infty\le z\le\infty,\ \alpha\succ 0,\ \theta\succ 0 \qquad (4) $$

This is new form of general density function of *SUUR* estimator with mean zero and variance equal $\frac{(\alpha-1)}{\theta^2\left(\alpha-2\right)}$. So that the variance is exited when $\alpha\neq 2,$ and $\theta\succ 0$. For $\alpha=n$, and $\theta=1$, we get the mean, the variance, and the probability density function of *Zellner's* as special case. The plots of the density function as given in (4) for α = 5 and θ = 0.25, 0.5, 0.75, 1.0, 1.25, and 1.75 are presented in figures (1) and (2).

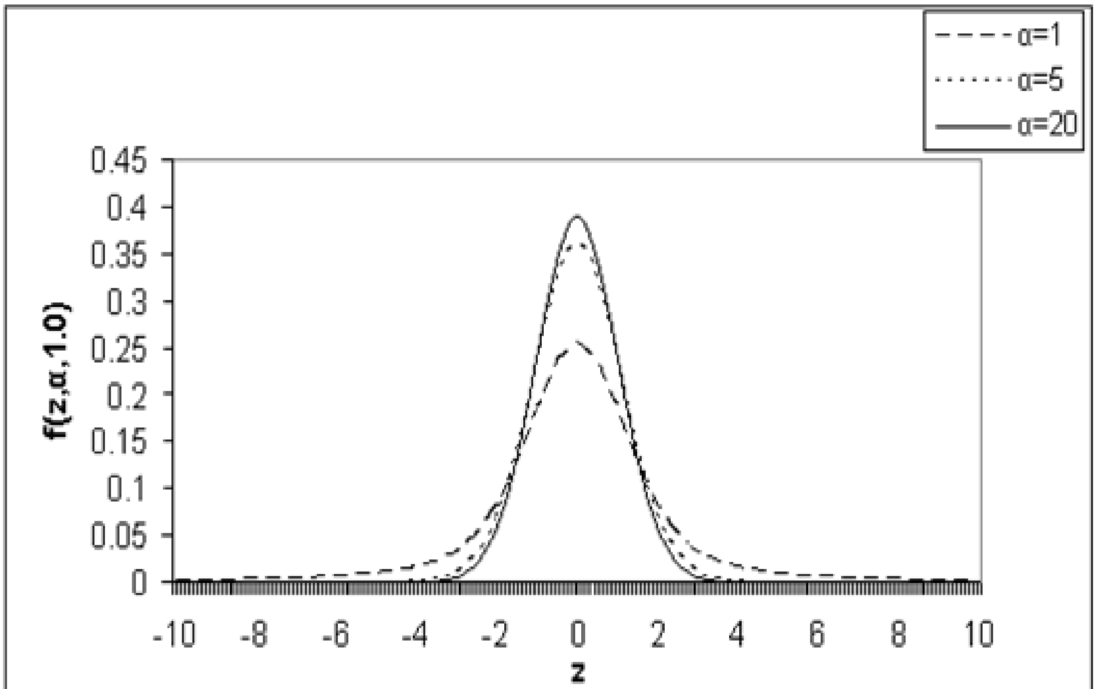


Figure(1): The probablility Density Function of Z for α=5 and θ =0.25, 0.5, 0.75



Figure(2): The Probablility Density Function of Z for α=5 and θ= 1, 1.25, 1.75

We may be noted from the figures that the degree of skewness of the distribution is symmetric around zero. With α =5, the flatness of the probability density function is observed to be an increasing when the values of θ is decreasing. So that, we expected that the variance of this distribution is a function of θ. The measure of kurtosis can be considered as a function of the parameter α as shown in figure (3). With θ =1, we observed to have less flatness as the parameter α increased.



Figure(3): The Probablility Density Function of Z for α=1, 5, 20 and θ= 1.0

3. The Properties Distribution for SURE Estimated

The cumulative distribution function, and the r $^{\underline{th}}$ moment, the characteristics function for SUUR equations estimators are derived in this section.

The ***Cumulative distribution*** function of (4) is

$$F(X) = \int_{-\infty}^{X} f(z, \alpha, \theta) dz$$

$$= C_1 . \int_0^1 t^{\frac{1}{2}-1} (1-t)^{\frac{\alpha+1}{2}-1} . \int_{-\infty}^{X} e^{-\frac{(\theta^2 (1-t))}{2} Z^2} dz\, dt$$

$$= C_1 \frac{\sqrt{2\pi}}{\theta} . \int_0^1 t^{\frac{1}{2}-1} (1-t)^{\frac{\alpha}{2}-1} . \phi\left(\theta \sqrt{(1-t)} X\right) dt$$

$$= \frac{1}{\beta\left(\frac{1}{2}, \frac{\alpha}{2}\right)} \int_0^1 t^{\frac{1}{2}-1} (1-t)^{\frac{\alpha}{2}-1} . \phi\left(\theta \sqrt{(1-t)} X\right) dt \tag{5}$$

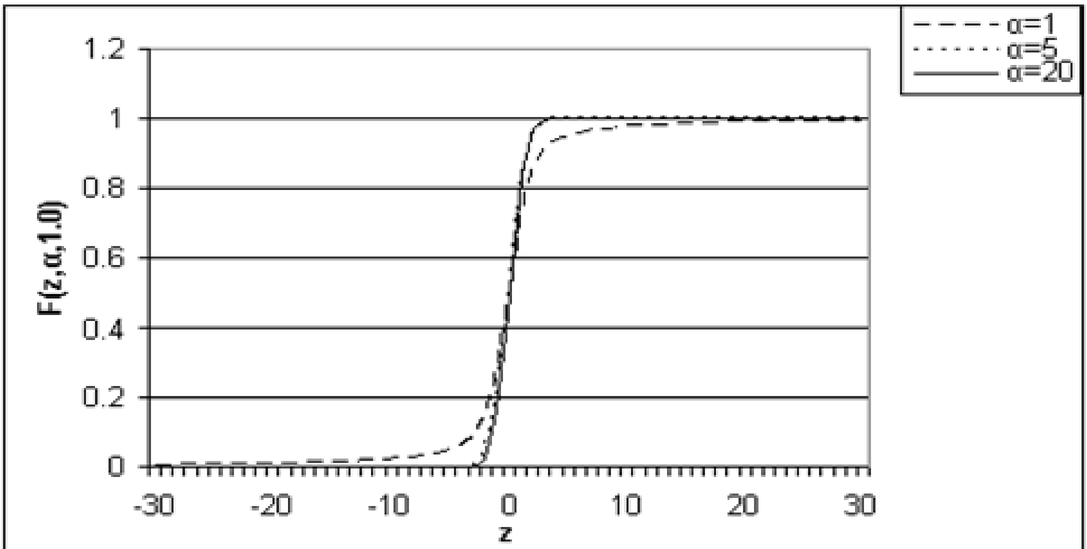
where

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{X} e^{-\frac{1}{2} Z^2} dz$$

and

$$c_1 = c \frac{\Gamma\left(\frac{\alpha+2}{2}\right)}{\Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{\alpha+1}{2}\right)}$$

It is interesting to note, from (5), that the cumulative distribution is the integration from zero to one for the beta distribution multiply by the cumulative distribution of a standard normal. Also, we have $F(-\infty) = 0$, $F(0) = 0.5$, and $F(\infty) = 1$. Figure (4) illustrated the plots of the cumulative distribution of Z when the parameter α =1, 5, 20 and θ =1.



Figure(4): The Cumulative Distribution Function for α=1, 5, 20 and θ= 1.0

To derive the ***r^th^ moment*** of *Z*, we found that $E(Z^{2r+1})=0,\ r=0,1,2,\ldots$ because it is odd function. So we will obtain the 2rth moment of *Z* as follows:

$$E\left(Z^{2r}\right)=\int_{-\infty}^{+\infty} Z^{2r} f\,(Z,\alpha,\theta)dZ\,.$$

$$=\frac{\theta\sqrt{2\pi}}{\alpha}\left[\frac{\left(\Gamma\frac{\alpha+1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{\alpha}{2}\right)}\right]^2 \cdot \frac{\Gamma\left(\frac{\alpha+2}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{\alpha+1}{2}\right)}\int_{-\infty}^{+\infty} z^{2r} e^{\frac{(\theta z)^2}{2}}\int_0^1 t^{\frac{1}{2}-1}(1-t)^{\frac{\alpha+1}{2}-1} e^{\frac{(\theta z)^2 t}{2}}\,dtdz$$

$$=2^r\,\frac{\Gamma\left(\frac{\alpha+1}{2}\right)\Gamma\left(\frac{\alpha}{2}-r\right)\Gamma\left(r+\frac{1}{2}\right)}{\theta^{2r}\ \Gamma\left(\frac{\alpha+1}{2}-r\right)\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(\frac{1}{2}\right)},\qquad (r=1,2,\cdots). \tag{6}$$

The fourth moment about the mean is used as a measure of Kurtosis, $\kappa=3+\frac{6}{(\alpha-1)(\alpha-4)}$, which is the degree of flatness of the density near its center. The largest degree of flatness was satisfied when $\alpha=1$, or 4, but if we have $\alpha=\infty$, we will get the smallest flatness of the probability function. This is agreeing with Figure (3). The first and the second moment of *Z* are

$$E\,(\,z\,)\,=\,0, \tag{7}$$

and

$$E\left(z^2\right)=\frac{(\alpha-1)}{\theta^2(\alpha-2)}. \tag{8}$$

We can obtained the mean and variance of *Zellner*(1962, 1972) as special case from our results in (7) and (8) when $\alpha=n$, and $\theta=1$.

The ***characteristic function*** of Z can be derived and expanded up to any order as follows:

$$\Phi_z(\tau)=E\left(e^{i\tau z}\right)$$

$$=c_1\int_{-\infty}^{+\infty} e^{i\tau z} e^{-\frac{(\theta z)^2}{2}}\int_0^1 t^{\frac{1}{2}-1}(1-t)^{\frac{\alpha+1}{2}-1} e^{\frac{(\theta z)^2 t}{2}}\,dtdz$$

$$=c_1\int_0^1 t^{\frac{1}{2}-1}(1-t)^{\frac{\alpha+1}{2}-1}\int_{-\infty}^{+\infty} e^{-\frac{(1-t)\theta^2}{2}\left(z-\frac{i\tau}{(1-t)\theta^2}\right)^2+\frac{(i\tau)^2}{2(1-t)\theta^2}}\,dzdt$$

$$=c_1\frac{\sqrt{2\pi}}{\theta}\int_0^1 t^{\frac{1}{2}-1}(1-t)^{\frac{\alpha}{2}-1} e^{-\frac{\tau^2}{2(1-t)\theta^2}}\,dt\,,$$

Then

$$\Phi_z(\tau)=\frac{\Gamma\left(\frac{\alpha+1}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)}\sum_{r=0}^{\infty}\frac{1}{r!}\frac{\Gamma\left(\frac{\alpha}{2}-r\right)}{\Gamma\left(\frac{\alpha+1}{2}-r\right)}\left(-\frac{\tau^2}{2\theta^2}\right)^r, \tag{9}$$

It is easily to prove that $\Phi_Z(0)=1$. We can be obtained another form of the characteristic function as follows:

$$\begin{aligned}\Phi_z(\tau)&=\sum_{r=0}^{\infty}\frac{1}{r!}\frac{\left(\frac{1-\alpha}{2}\right)_r}{\left(\frac{2-\alpha}{2}\right)_r}\left(-\frac{\tau^2}{2\theta^2}\right)^r\\&={}_1F_1\left(-\frac{\alpha-1}{2};-\frac{\alpha-2}{2};-\frac{\tau^2}{2\theta^2}\right)\\&=e^{-\frac{\tau^2}{2\theta^2}}{}_1F_1\left(\frac{1}{2};-\frac{\alpha-2}{2};\frac{\tau^2}{2\theta^2}\right)\\&=e^{-\frac{\tau^2}{2\theta^2}}\sum_{r=0}^{\infty}\frac{1}{r!}\frac{\left(\frac{1}{2}\right)_r}{\left(-\frac{\alpha-2}{2}\right)_r}\left(\frac{\tau^2}{2\theta^2}\right)^r.\end{aligned} \tag{10}$$

When α tends to infinity the characteristic function of (9) will be:

$$\lim_{\alpha\to\infty}\Phi_z(\tau)=\sum_{r=0}^{\infty}\frac{1}{r!}\left(-\frac{\tau^2}{2\theta^2}\right)^r\lim_{\alpha\to\infty}\frac{\Gamma\left(\frac{\alpha}{2}-r\right)}{\Gamma\left(\frac{\alpha+1}{2}-r\right)}\frac{\Gamma\left(\frac{\alpha+1}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)}. \tag{11}$$

Since

$$\lim_{p\to\infty}\frac{\Gamma(p+h)}{\Gamma p}=p^h,$$

Then, we have

$$\begin{aligned}\lim_{\alpha\to\infty}\Phi_z(\tau)&=\sum_{r=0}^{\infty}\frac{1}{r!}\left(-\frac{\tau^2}{2\theta^2}\right)^r\\&=e^{-\frac{1}{2\theta^2}\tau^2}.\end{aligned} \tag{12}$$

This is the characteristic function of the normal distribution with mean zero and variance θ^2. Then, the probability density function, as given in (4), tends to the standard normal distribution when $\theta=1$, and α tends to infinity.

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