A few relations for sums of differences of discrete probability distributions

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In this article some interesting relationship between sums of differences and different moments of discrete probability distributions with support {0,1,2,…} have been derived. The work has been motivated by the list of such relationships for Generalized Poisson distribution (GPD) presented in Consul (1989 page 66-67). Here some generalized relationships have been obtained between sums (Σ) of difference [both forward (Δ) and backward (∇)] of probabilities and moments for any discrete distribution having support {0,1,2,…} and all the corresponding results of Consul (1989) for GPD have been seen as particular cases of one of the general relations. Some new relations between different partial sums of probabilities with different moments for discrete distributions have also been stated.

Key words: - *Generalized Poisson Distribution; Shift Operator; Forward and Backward Difference Operators; Discrete Probability Distribution; Stirling number of the* 1^{st}, 2^{nd}, 3^{rd} *and* 4^{th} *kind; Binomial coefficients.*

1. INTRODUCTION.

Importance of finite difference operators, factorial expressions, Stirling numbers in the study of discrete probability distributions and their moment properties is well established. Use of finite difference in the context of the derivation of expressions for moments of discrete probability distributions had been discussed in a series of articles in American Statistician during 1981-1983 by Link (1981), Johnson & Kotz (1981), Chan (1982), Rao & Janardan (1982) and Khatri (1983). Consul (1989, pages 66-67) in his book entitled "Generalize Poisson distribution" listed some relationship between difference sums of the probabilities of his generalized Poisson distribution (GPD) and its different moments. In this article in section 2. some interesting relationship between the sums of differences and moments of discrete probability distribution have been derived which hold for any discrete distribution with support {0,1,2,...}. While in section 3. some relationships between partial sums of probabilities and moments for discrete distributions with finite support have been presented.

Before presenting the main results of the article, we provide the various essential notations, definitions and relations regarding the different finite difference operators, Stirling numbers, factorial expressions etc. used in this work.

1. The forward difference operator Δ is defined as $\Delta f(x) = f(x+1) - f(x)$
2. The backward difference ∇ is defined as $\nabla f(x) = f(x) - f(x-1)$
3. The displacement operator E is given by $E f(x) = f(x+1)$
4. The symbolic relationship between E and Δ is $E \equiv \Delta -1$
5. $\Delta^n f(x) = \sum_{j=0}^{n} \binom{n}{j} (-1)^j f(x+n-j)$
6. $E^n f(x) = \sum_{j=0}^{n} \binom{n}{j} \Delta^j f(x)$
7. $x^{(n)} = x(x-1)(x-2)\dots(x-n+1)$
8. $x^{[n]} = x(x+1)(x+2)\dots(x+n-1)$
9. $x^{(n)} = \sum_{j=0}^{n} x^j S_1(n, j)$
10. $x^n = \sum_{j=0}^{n} x^{(j)} S_2(n, j)$
11. $x^{[n]} = \sum_{j=0}^{n} x^j S_3(n, j)$
12. $x^n = \sum_{j=0}^{n} x^{[j]} S_4(n, j)$

Where $S_1(.\,,\,.)$, $S_2(.\,,\,.)$, $S_3(.\,,\,.)$ and $S_4(.\,,\,.)$ are respectively the Stirling 1st, 2nd 3rd and 4th kind respectively (Johnson et al (1992) pages 8-11 and Johnson et al (1997) page 8)

For a given DISCRETE probability distribution $\{p_0, p_1, p_2, \dots\}$ we have the following definitions and relations of moments.

13. $\mu'_r = \sum_{x\geq 0} x^r p_x$
14. $\mu'_{[r]} = \sum_{x\geq 0} x^{[r]} p_x$
15. $\mu'_{(r)} = \sum_{x\geq 0} x^{(r)} p_x$
16. $\sum_{j=0}^{r} S_3(r,j) \sum_{i\geq 0} i^j p_i = \mu'_{[r]}$
17. $\sum_{j=0}^{r} S_4(r,j) \sum_{i\geq 0} i^{[j]} p_i = \mu'_r$

18. $\sum_{j=0}^{r} S_1(r,j) \sum_{i\geq 0} i^j \ p_i = \mu'_{(r)}$

19. $\sum_{j=0}^{r} S_2(r,j) \sum_{i\geq 0} i^{(j)} \ p_i = \mu'_r$

2. RELATIONS BETWEEN SUMS OF DIFFERENCES OF PROBABILITIES AND MOMENTS.

Theorem 1. $\sum_{x\geq 0} x^n \nabla^m p_x = (-1)^m m! [S_2(.\,,m) + \mu'_.]^n$, where on expansion $[S_2(.\,,m)]^k = S_2(k,m)$ and $(\mu'_.)^k = \mu'_k$ and $S_2(k,m)$ is the Stirling number of the second kind.

For $n<m$ as $S_2(n,m) = 0$, therefore

$$\sum_{x\geq 0} x^n \nabla^m p_x = 0.$$

Proof:- $\sum_{x\geq 0} x^n \nabla^m p_x$

$$= \sum_{x\geq 0} x^n (1-E^{-1})^m p_x$$

$$= \sum_{x\geq 0} x^n (-1)^m (E^{-1}-1)^m p_x$$

$$= \sum_{x\geq 0} x^n (-1)^m \sum_{i=0}^{m} (-1)^{m-i} \binom{m}{i} E^{-i} p_x$$

$$= \sum_{x\geq 0} x^n (-1)^m \sum_{i=0}^{m} (-1)^{m-i} \binom{m}{i} p_{x-i}$$

$$= \sum_{i=0}^{m} (-1)^{2m-i} \binom{m}{i} \sum_{x\geq i} x^n \ p_{x-i}$$

[since $p_x=0$ for $x<0$]

$$= \sum_{i=0}^{m} (-1)^{2m-i} \binom{m}{i} \sum_{x\geq 0} (x+i)^n p_x$$

[on replacing x by x+i i.e.as $x \leftarrow x+i$]

$$= \sum_{i=0}^{m} (-1)^{2m-i} \binom{m}{i} \sum_{x\geq 0} [\sum_{r=0}^{n} \binom{n}{r} x^{n-r} i^r]\, p_x$$

$$= (-1)^m m! \sum_{r=0}^{n} \binom{n}{r} \{\frac{1}{m!} \sum_{i=0}^{m} (-1)^{m-i} \binom{m}{i} i^r\} \{\sum_{x\geq 0} x^{n-r} \ p_x\}$$

$$= (-1)^m m! \sum_{r=0}^{n} \binom{n}{r} S_2(r,m) \ \mu'_{n-r}$$

[since $S_2(r,m) = \frac{1}{m!} \sum_{i=0}^{m} (-1)^{m-i} \binom{m}{i} i^r$]

$$= (-1)^m m! [S_2(.,m) + \mu'_.]^n .$$

The above theorem can be used along with the values given below of the Stirling number of the 2nd kind to obtain a large number of relationships.

Table 1. *Table of values of $S_2(n, m)$ for various values of n, m*

n	m					
	1	2	3	4	5	6
1	1					
2	1	1				
3	1	3	1			
4	1	7	6	1		
5	1	15	25	10	1	
6	1	31	90	65	15	1

In fact by choosing different values for n and m all the relations listed in Consul (1989, pp. 66-67) for GPD can be easily derived. Two examples are given below to illustrate the application of the Theorem 1.

Example 1.

$$\sum_{x\geq 0} x^3\ \nabla^3\ p_x = (-1)^3\ 3!\ [S_2(.\ ,3) + \mu'_.]^3$$

$$= -6[S_2(3\ ,3) + 3\ S_2(2\ ,3)\ \mu'_1 + 3\ S_2(1\ ,3)\ \mu'_2 + S_2(0\ ,3)\ \mu'_3]$$

$$= -6\ .\ 1 = -6$$

Example 2.

$$\sum_{x\geq 0} x^3\ \nabla^2\ p_x = (-1)^2\ 2!\ [S_2(.\ ,2) + \mu'_.]^3$$

$$= 2[S_2(3\ ,2) + 3\ S_2(2\ ,2)\ \mu'_1 + 3\ S_2(1\ ,2)\ \mu'_2 + S_2(0\ ,2)\ \mu'_3]$$

$$= 2[3 + 3\ .\ 1\ .\ \mu'_1]$$

$$= 6 + 6\ \mu'_1$$

It has been observed that the relations for the GPD listed in Consul (1989) are not special characteristics of GPD, all these relations hold good for any discrete distribution having support {0,1,2,…}.

CORROLARY 1.

(a) $$\sum_{x\geq 0} x^n\ \nabla^{n-1}\ p_x = (-1)^{n-1}\ (n-1)!\ [S_2\ (n,n-1) + n\ \mu'_1]$$

(b) $$\sum_{x\geq 0} x^n\ \nabla^{n-2}\ p_x = (-1)^{n-2}\ (n-2)!\ \left[S_2\ (n,\ n-2) + n\ S_2\ (n-1,n-2)\ \mu'_1 + \frac{n(n-1)}{2}\ \mu'_2\right]$$

(c) $$\sum_{x\geq 0} x^n\ \nabla^n\ p_x = (-1)^{n-1}\ n!$$

It is therefore obvious that the corresponding relations for Poisson probabilities (Arion, 1937) too can be easily derived from the Theorem 1.

Theorem 2. $$\sum_{x\geq 0} x^{(n)}\ \Delta^m\ p_x = m!\ [\mu'_{(.)} - \beta_{.\ m}]^n - \sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}\sum_{x=0}^{\max(0,\,j-1)}(x-j)^{(n)}\ p_x\ ,$$

where on expansion $(\mu'_{(.)})^k = \mu'_{(k)}$ and $(\beta_{.\ m})^k = \beta_{k\ m}$ and $\beta_{r\ m} = \sum_{k=0}^{r} S_3(r,\ k)\ S_2\ (k,\ m)$,

wherein S_3 (n, m) is the Stirling number of the third kind.

Proof :- $$\sum_{x\geq 0} x^{(n)}\ \Delta^m\ p_x$$

$$= \sum_{x\geq 0} x^{(n)}\ (E-1)^m\ p_x$$

$$= \sum_{x\geq 0} x^{(n)} \sum_{j=0}^{m} (-1)^{m-j}\binom{m}{j} E^j\ p_x$$

$$= \sum_{j=0}^{m} (-1)^{m-j}\binom{m}{j}\sum_{x\geq 0} x^{(n)}\ p_{x+j}$$

$$= \sum_{j=0}^{m} (-1)^{m-j}\binom{m}{j}\sum_{x\geq j} (x-j)^{(n)}\ p_x$$

$$= \sum_{j=0}^{m} (-1)^{m-j}\binom{m}{j}\left[\sum_{x\geq 0} (x-j)^{(n)}\ p_x - \sum_{x=0}^{\max(0,\,j-1)} (x-j)^{(n)}\ p_x\right]$$

$$= \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} [\sum_{x\ge 0} \{\sum_{r=0}^{n} (-1)^{r} \binom{n}{r} j^{[r]} x^{(n-r)}\} p_x$$

$$- \sum_{x=0}^{\max(0,j-1)} (x-j)^{(n)} p_x]$$

$$= \sum_{r=0}^{n} (-1)^{r} \binom{n}{r} \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} \{ \sum_{k\ge 0}^{r} S_3(r,k)\, j^k\}$$

$$\sum_{x\ge 0} x^{(n-r)} p_x - \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} \sum_{x=0}^{\max(0,j-1)} (x-j)^{(n)} p_x$$

$$= \sum_{r=0}^{n} (-1)^{r} \binom{n}{r} \sum_{k\ge 0}^{r} S_3(r,k) \{\sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} j^k\}$$

$$\sum_{x\ge 0} x^{(n-r)} p_x - \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} \sum_{x=0}^{\max(0,j-1)} (x-j)^{(n)} p_x$$

$$= m! \sum_{r=0}^{n} (-1)^{r} \binom{n}{r} \{ \sum_{k\ge 0}^{r} S_3(r,k) S_2(k,m)\}\ \mu'_{(n-r)} -$$

$$\sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} \sum_{x=0}^{\max(0,j-1)} (x-j)^{(n)} p_x$$

$$= m! \sum_{r=0}^{n} (-1)^{r} \binom{n}{r} \beta_{r\,m}\ \mu'_{(n-r)} - \sum_{j=0}^{m} (-1)^{m-j}$$

$$\binom{m}{j} \sum_{x=0}^{\max(0,j-1)} (x-j)^{(n)} p_x \,.$$

CORROLLARY 2.

$$\sum_{x\ge 0} x^n\ \Delta^m\ p_x = m! \sum_{j=0}^{n} S_2(n,j)\, [\mu'_{(.)} - \beta_{.\,m}]^j -$$

$$\sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} \sum_{x=0}^{\max(0,k-1)} (x-k)^n p_x$$

The above theorem can be used along with the values of $\beta_{r\ \ m}$ given below to obtain a large number of relationships.

Table 2. *Table of values of* $\beta_{r\ \ m}$ *for various values of n, m*

n	m					
	1	2	3	4	5	6
1	1					
2	2	1				
3	6	6	1			
4	24	36	12	1		
5	120	240	120	20	1	
6	720	1800	1200	300	30	1

Two illustrative examples of the application of Theorem 2 have been given below.

Example 3.

$$\sum_{x\ge 0} x^{(2)}\ \Delta\ p_x$$

$$= 1![\mu'_{(.)} - \beta_{.\,1}]^2 - \sum_{j=0}^{1} (-1)^{1-j} \binom{1}{j} \sum_{x=0}^{\max(0,j-1)} (x-j)^{(2)} p_x$$

$$= 1!\ [\mu'_{(2)}\ \beta_{0\,1} - 2\mu'_{(1)} \beta_{1\,1} + \beta_{2\,1}]$$

$$-\quad [1.\ (-1)^{(2)}\ p_o]$$

$$= \mu'_{(2)}\ .\ 0\ - 2\mu'_{(1)}\ .\ 1\ + 2 - 2\, p_o$$

$$= -2\mu'_{(1)} - 2\, p_o + 2$$

Example 4.

$$\sum_{x\ge 0} x^{(2)}\ \Delta^2\ p_x$$

$$= 2!\ [\mu'_{(.)} - \beta_{.\,1}]^2 - \sum_{j=0}^{2} (-1)^{2-j} \binom{2}{j} \sum_{x=0}^{\max(0,j-1)}$$

$$(x-j)^{(2)} p_x$$

$= 2! \, [\, \mu'_{(2)} \, \beta_{0\,2} - 2\, \mu'_{(1)} \beta_{1\,2} + \beta_{2\,2}] - (-1)^{2-1} \,.\, 2 \,.$

$(-1)\,(-1-1)\, p_o - (-1)^{2-2} \,.\, 1 \,.\, [(-2)(-2-1)\, p_o + (-1)$

$(-1-1)p_1]$

$= -\,2\, p_o - 2\, p_1 + 2$

3. RELATIONS BETWEEN PARTIAL SUMS OF PROBABILITIES AND MOMENTS.

For a given DISCRETE probability distribution $\{p_0, p_1, p_2, \ldots, p_n\}$ the following relationships hold.

1. $\sum_{j=0}^{n} F_j = \mu'_1$

2. $\sum_{j=0}^{n} j^{[r-1]} F_j = \frac{1}{r} \mu'_{[r]}$

3. $\sum_{j=0}^{n} j^{(r-1)} F_j = \frac{1}{r} \mu'_{(r)}$

4. $\sum_{j=0}^{n} \binom{n}{j} F_j = \sum_{i=0}^{n} 2^{i} p_i$

5. $\sum_{j=0}^{n} j \binom{n}{j} F_j = \sum_{i=0}^{n} (i\, 2^{\,i-1})\, p_i$

6. $\sum_{j=0}^{n} \binom{n}{j}^2 F_j = \sum_{i=0}^{n} \binom{2i}{i} p_i$

7. $\sum_{j=0}^{r-1} \frac{(-1)^j}{j!(r-j-1)!} \sum_{i=0}^{n} \frac{p_i}{i+j+1} = \sum_{i=0}^{n} \frac{1}{(i+1)^{[r]}} p_i \,.$

Where $F_j = p_j + p_{j+1} + \cdots + p_n$.

The above results can be used as alternative formulas for computing different moments. It may be noted that the last result i.e. Number 7. gives a inverse factorial moment.

Note: - For distributions with finite support say, {0,1,2,…,N} Theorem 2 holds while Theorem 1 will have to be adjusted properly before application. In such cases incomplete moments will come in to picture.

Two tables of the values of Stirling numbers of 3rd and 4th kind (Johnson et al., 1997) have been provided below.

Table 3. *Table of values of S_3 (n, m) for various values of n, m*

n	m					
	1	2	3	4	5	6
1	1					
2	1	1				
3	2	3	1			
4	6	11	6	1		
5	24	50	35	10	1	
6	120	274	225	85	15	1

Table 4. *Table of values of S_4 (n, m) for various values of n, m*

n	m					
	1	2	3	4	5	6
1	1					
2	-1	1				
3	1	-3	1			
4	-1	7	-6	1		
5	1	-15	25	-10	1	
6	-1	31	-90	65	-15	1

REFRENCES

1. Arion L. A. (1937) **Type B Gram Charlier Series,** Ann. Math. Statist: 183-192.

2. Link Robert F. (1981) **Moments of Discrete Probability Distributions Derived Using Finite Difference Operators,** American Statistician: Vol.: 35, No. 1, 44-46.

3. Johnson N. L. and Kotz, S. (1981) **Moments of Discrete Probability Distributions Derived Using Finite Difference Operators,** American Statistician: Vol.: 35, No. 4, 268.

4. Chan Beda (1982) **Derivation of Moment Formulas by Operator Valued Probability Generating Functions**, American Statistician: Vol.: 36, No. 3, 179-181.

5. Rao B. Raja and Janardan K. G. (1982) **On the Moments of Multivariate Discrete Distributions Using Finite Difference Operators**, American Statistician: Vol.: 36, 381-383.

6. Khatri C. G. (1983) **Some Remarks On the Moments of Discrete Distributions Using Difference Operators,** American Statistician: Vol.: 37, 96-97.

7. Consul P.C. (1989) **Generalized Poisson Distributions,** *Marcel Dekker* , NY

8. Johnson N. L. , Kotz, S. and Kemp, A. W. (1992) **Univariate Discrete Distributions**, 2nd Edition, John Wiley, NY

9. Johnson N. L. , Kotz, S. and Balakrishnan, N.(1997) **Discrete Multivariate Distributions**, John Wiley, NY