PREDICTION BOUNDS FOR THE WEIBULL DISTRIBUTION

A. A. Jayawardhana

Department of Mathematics

Pittsburg State University

Pittsburg, KS 66762

and

V. A. Samaranayake

Department of Mathematics and Statistics

University of Missouri-Rolla

Rolla, MO 65409-0020

ABSTRACT

An approximate method for constructing prediction limits for a future Weibull observation, based on the Maximum Likelihood Predictive Density approach proposed by Lejeune and Faulkenberry (1982), is derived. It is proposed as a simple and versatile alternative to existing methods, with prediction bounds given by easily computable closed form expressions. The proposed procedure can be adapted to obtaining both upper and lower prediction bounds under either Type I or Type II censoring schemes. Results of a Monte Carlo simulation study that compares the coverage probabilities of the proposed method and several modified versions are given. It is shown that the proposed method provide close to nominal coverage probabilities in many instances and the modified versions can be used in its place in situations where its coverage is liberal.

Key words: Prediction Limits, Weibull Distribution, Profile Likelihood, Maximum Likelihood Predictive Density, Type I censoring, Type II censoring

1. INTRODUCTION

Prediction of an unobserved random variable is a fundamental problem in statistics. Patel (1989) provides an extensive survey of literature on this topic. In the areas of reliability and life-testing, this problem translates to obtaining prediction intervals for life distributions such as the Exponential and the Weibull. For the Weibull case, several authors have addressed this issue as well the more complicated problem of deriving prediction bounds for order statistics from a future sample. These include Mann and Saunders (1969), Mann (1970), Antle and Rademaker (1972), Lawless (1973), Mann, Schafer and Singpurwalla (1974), Mann (1976), Engelhardt and Bain (1979, 1982), and Fertig, Mayer and Mann (1980). While these methods have their strengths, a simple procedure that can be utilized by non-statisticians, and which provides easily computable explicit expressions for both lower and upper prediction bounds is currently unavailable. Our objective is to address this need by proposing a relatively simple, approximate, non-Bayesian solution to the Weibull prediction problem.

It is a rule rather than an exception that most industrial use of statistical methods are carried out by non-statisticians. This is especially true in small to medium sized companies, having no formally trained statisticians. Professionals in such organizations tend to favor simple statistical methods that are easy to use and quick to carry out, over sophisticated statistical methods that require a deeper conceptual understanding or the use of extensive numerical computations, unless of course the method of interest in implemented in a readily available computer software package. We proposed such a simple method for the problem of obtaining prediction intervals. As will be shown via a simulation study, the proposed method and several variations of it produce lower and upper prediction bounds with coverage probabilities very close to the nominal value. We will also show how the proposed method can be adapted to work with both Type I and Type II censored data. If the simple estimates given in Jayawardhana and

Samaranayake (2003) are used for parameter estimation, then this method does not involve any numerical integration or trial-and-error procedures. If the maximum likelihood estimates are used for parameter estimation, then the proposed method requires finding these estimates iteratively. As shown later, the use of the simple estimators suffice for most cases. It does, however, require the use of unbiasing constants published in Jayawardhana and Samaranayake (2003) in either situation.

In brief, the proposed method is based on the estimation of a predictive density whose percentile points can be expressed explicitly. We proposed the use of these percentile points as our prediction bounds. Simulation results show that in most cases the proposed prediction bounds yield coverage probabilities very close to the nominal values. However, there are some situations where they yielded slightly liberal coverage. Several modifications to the method are introduced to provide better coverage in such instances. These variations as well as the original bounds are then compared using a Monte Carlo Simulation study.

A brief summary of past literature is given in the next section. It is followed by a section on the maximum likelihood predictive density approach. Sections on the derivation of prediction bounds for the Type II and Type I censored cases follow. Two examples are given next, followed by a section on the Monte Carlo simulation study. The last two sections discuss the simulation results and summarize the conclusions.

2. A BRIEF REVIEW OF LITERATURE

One of the earlier works on prediction for the Weibull distribution is by Mann and Saunders (1969). They considered prediction intervals for the smallest of a set of future observations, based on a small (two or three) preliminary sample of past observations. An expression for the warranty period (time before the failure of the first ordered observation from a set of future observations or a lot) was derived as a function of the ordered past observations. Mann (1970) extended the results for lot sizes n = 10 (5) 25 and sample sizes m = 2 (1) $n-3$ for a specified assurance level of 0.95. This method requires numerical integration. In addition, the tables provided are limited to sample sizes less than 25 and are given only for the assurance level of 0.95.

Antle and Rademaker (1972) provided a method of obtaining a prediction bound for the largest observation from a future sample of the Type I extreme value distribution, based on the maximum likelihood estimates of the parameters. They used Monte Carlo simulations to obtain the prediction intervals. Using the well-known relationship between the Weibull distribution and the Type I extreme value distribution one can use their method to construct an upper prediction limit for the largest among a set of future Weibull observations. However this method is valid only for complete samples and limited to constructing an upper prediction limit for the largest among a set of future observations.

Lawless (1973) proposed a method for constructing prediction intervals for the smallest ordered observation among a set of k future observations based on a Type II censored sample of past observations. These results are based on the conditional distribution of the maximum likelihood estimates given a set of ancillary statistics. This procedure is exact, but it requires numerical integration, for each new sample obtained, to determine the prediction bounds.

Mann, Schafer and Singpurwalla (1974) and Mann (1976) derived a method for constructing prediction intervals for the k^{th} smallest failure time of a lot of size n, using the first r order statistics $(r < k)$. Their approach is based on an approximation to Fisher's F distribution using the results by Pyke (1965), who discussed the distribution of the differences of extreme value order statistics. This method works for Type II censored data and requires the use of a table giving weights for obtaining best linear invariant estimators and mean squared errors associated with these estimators. It also requires the use of tables (see Mann (1968), Table 3.11) of variances of $Z_r = (X_r - \eta)/\xi$ and $Z_k = (X_k - \eta)/\xi$, and covariance between them, where X_r, and X_k are natural logarithm of the r^{th} and k^{th} order statistics, with η and ξ denoting the natural logarithm of the scale parameter and the inverse of the shape parameter of the underlying Weibull distribution, respectively. They suggest a simplified version for sample sizes of 100 or more.

Engelhardt and Bain (1979) used the log-χ^2 approximation derived by Engelhardt and Bain (1977) to construct prediction intervals, based on Type II censored data, for the smallest ordered observation from a future set of observations from the Type I extreme value distribution. They also extended their method to obtaining approximate prediction intervals for the jth smallest of k future observations. This method uses the simplified linear unbiased estimates of location and scale parameters of the distribution and requires the use of tabulated values (available in Engelhardt and Bain (1977)) of variances and covariances of these estimators. It also depends on obtaining percentile points of the distribution of $T = (\hat{u} - y_{(1)}) \big/ \hat{b}$, where $\hat{u}$ and $\hat{b}$ are the location and scale parameters and $y_{(1)}$ is the minimum of a set of k future observations of an extreme value distribution of Type I. Since these percentile points cannot be found explicitly, the authors suggest employing either an iterative method or a trial-and-error process in finding an approximation to these percentile points.

Fertig et al. (1980) provided a method to find lower and upper prediction limits for a future observation from a Weibull distribution. They provided a table of percentiles of the distribution of $S = (\tilde{u} - y_{(1)}) \big/ \tilde{b}$ for r = 3, 5 (5) n and n = 5 (5) 25, where $\tilde{u}$ and $\tilde{b}$ are the best linear invariant estimates of location and scale parameters of an extreme value distribution and $y_{(1)}$ is the smallest ordered observation of a future sample of size m from the same distribution.

These Monte Carlo simulation based percentiles were computed only for the case $m = 1$. They also provided an analytical approximation, that can be used for other combinations of r and n not given in their table. This latter method uses numerical integration to determine the percentile points and is recommended for use when sample size is greater than 25. This approximate method, however, can be applied not only for the case $m = 1$, but also for the $m > 1$ case. Both of the above methods are valid only for the Type II censored data.

In a follow-up to their 1979 paper, Engelhardt and Bain (1982) provided three simplified approximations, F_1, F_2 , F_3, to the distribution of $T = (\hat{u} - y_{(1)}) / \hat{b}$ defined earlier in this section. While F_1 still requires an iterative method for determining the percentile points, the other two admit explicit expressions for the percentiles. Approximations F_1 and F_2 require the use of both tables of unbiasing constants (for the linear estimates of the parameters) and tables of variances and covariances of the standardized estimates. Approximation F_3 requires the use of the use of the table of unbiasing constants only. The simplest of the three procedures, F_3, is based on a large sample approximation, and is recommended only for confidence levels between 50% and 90%. Methods based F_2 and F_3 are not recommended for constructing upper prediction limits. The authors state that approximation F_1 is “adequate, for most practical purposes, with upper prediction limits provided that the amount of censoring is moderate”. All three methods work under Type II censoring.

Mee and Kushary (1994) provided a simulation based procedure for constructing prediction intervals for Weibull populations for Type II censored case. This procedure is based on maximum likelihood estimation and requires an iterative process to determine the percentile points.

3. MAXIMUM LIKELIHOOD PREDICTIVE DENSITY

Consider a set of past observations X_1, X_2, ..., X_n from a distribution $f(x;\theta)$. Supppose Y_1, Y_2, ..., Y_m, is a set of future observations from the same distribution, independent of $\tilde{X} = (X_1, X_2, \ldots, X_n)'$. Let Z be some statistic based on $\tilde{Y} = (Y_1, Y_2, \ldots, Y_m)'$. Suppose one wishes to obtain an estimate of the density of some statistic $Z = h(\tilde{Y})$ based on the observed value $\tilde{x}$ of $\tilde{X}$.

Lejeune and Faulkenberry (1982) proposed the use of the function $\hat{f}(z|\tilde{x}) = k(\tilde{x})\, Sup_{\theta\in\Theta} f(\tilde{x};\theta)\, g(z;\theta)$ as a "predictive probability density" of Z, where $f(\tilde{x};\theta)$ is the joint probability density function of the X's and $g(z;\theta)$ is the probability density function of the statistic Z, Θ is the parameter space of the unknown parameter θ, and $k(\tilde{x})$ is a normalizing function. In brief, they suggest replacing the parameter θ in likelihood equation by its maximum likelihood estimate (MLE), $\hat{\theta}$, based on the data $(\tilde{x}, z)$ and then multiplying by the normalizing function $k(\tilde{x})$. Note that $\hat{\theta}$ is based on both $\tilde{x}$ and z and not merely on the past data $\tilde{x}$. The probability density thus obtained is called the maximum likelihood predictive density (MLPD) of Z.

4. PREDICTION BOUNDS FOR A WEIBULL RANDOM VARIABLE - TYPE II CENSORED CASE

A Weibull random variable X with scale parameter θ and shape parameter β has distribution function

$$F_X(x) = 1 - \exp\left[-(x/\theta)^{\beta}\right],\ x > 0,\ \theta > 0, \text{ and } \beta > 0\ .$$

Now let X_1, X_2, ..., X_r be an ordered set of Type II censored past observations from a random sample of size n $(r \le n)$ from the above Weibull distribution. Let Z denote a future observation from the same distribution. Let $\tilde{X} = (X_1, X_2, \ldots, X_r)'$. Then the joint probability density of $\tilde{X}$ and Z is given by

$$f(x_1, x_2, \ldots, x_r, z; \theta, \beta) = (n!)\{(n-r)!\}^{-1} \prod_{i=1}^{r} \left[\beta\, \theta^{-\beta} x_i^{\beta-1} \exp\left\{-\left(x_i/\theta\right)^{\beta}\right\}\right] \times$$

$$\exp\left\{-(n-r)\left(x_r/\theta\right)^{\beta}\right\} \beta\theta^{-\beta} z^{\beta-1} \exp\left\{-(z/\theta)^{\beta}\right\}$$

$$= (n!)\left[(n-r)!\right]^{-1} \beta^{r+1} \theta^{-(r+1)\beta} \left\{\prod_{i=1}^{r} x_i^{\beta-1}\right\} z^{\beta-1} \times$$

$$\exp\left[-\left\{\left(\sum_{i=1}^{r} x_i^{\beta}\right) + (n-r)x_r^{\beta} + z^{\beta}\right\}\theta^{-\beta}\right]. \quad (4.1)$$

Therefore, the log likelihood function given observations x_1, x_2, ..., x_r, and z is

$$\ln\left\{f(x_1, x_2, \ldots, x_r, z; \theta, \beta)\right\} = \ln\left[(n!)\left\{(n-r)!\right\}^{-1}\right] + (r+1)\ln\beta - (r+1)\beta\ln\theta$$

$$+ (\beta-1)\sum_{i=1}^{r}\ln x_i + (\beta-1)\ln z - \left\{\sum_{i=1}^{r}\left(x_i/\theta\right)^{\beta}\right\}$$

$$- (n-r)\left(x_r/\theta\right)^{\beta} - \left(z/\theta\right)^{\beta}.$$

Taking partial derivatives with respect to θ and β, setting them to zero and solving, we obtain

$$-(r+1)\hat{\beta}\hat{\theta}^{-1} + \hat{\beta}\left\{\left(\sum_{i=1}^{r} x_i^{\hat{\beta}}\right) + (n-r)x_r^{\hat{\beta}} + z^{\hat{\beta}}\right\}\hat{\theta}^{-(\hat{\beta}+1)} = 0$$

and

$$(r+1)\hat{\beta}^{-1} - (r+1)\ln\hat{\theta} + \sum_{i=1}^{r}\ln x_i + \ln z - \sum_{i=1}^{r}\left(x_i/\hat{\theta}\right)^{\hat{\beta}} \ln\left(x_i/\hat{\theta}\right) - (n-r)\left(x_r/\hat{\theta}\right)^{\hat{\beta}} \ln\left(x_r/\hat{\theta}\right)$$

$$-\left(z/\hat{\theta}\right)^{\hat{\beta}} \ln\left(z/\hat{\theta}\right) = 0.$$

Then the maximum likelihood estimate $\hat{\theta}$, of θ based on x_1, x_2, ..., x_r, and z, is given by

$$\hat{\theta} = \left[(r+1)^{-1} \left\{ \left(\sum_{i=1}^{r} x_i^{\hat{\beta}} \right) + (n-r) x_r^{\hat{\beta}} + z^{\hat{\beta}} \right\} \right]^{1/\hat{\beta}}, \tag{4.2}$$

where $\hat{\beta}$ denotes the maximum likelihood estimate of β based on x_1, x_2, ..., x_r, and z. Unfortunately, an expression for $\hat{\beta}$ is not available in explicit form. For the time being, we will assume that β is known and replace $\hat{\beta}$ in (4.2) by β. Using the method proposed by Lejeune and Faulkenberry (1982) the predictive density function of Z given $X_i = x_i$, $i =$ 1, 2, ..., r, is obtained by replacing θ in (4.1) by $\hat{\theta}$ given in (4.2). This yields

$$\tilde{f}(z) = k(\tilde{x}) \left\{ \left(\sum_{i=1}^{r} x_i^{\beta} \right) + (n-r) x_r^{\beta} + z^{\beta} \right\}^{-(r+1)} z^{\beta-1}, \tag{4.3}$$

under the assumption that β is known. Since $\int_{o}^{\infty} \tilde{f}(z)dz = 1$, we have that

$$\int_0^{\infty} k(\tilde{x}) \left\{ \left(\sum_{i=1}^{r} x_i^{\beta} \right) + (n-r) x_r^{\beta} + z^{\beta} \right\}^{-(r+1)} z^{\beta-1} dz = 1. \tag{4.4}$$

Equation (4.4) implies

$$\left\{k\left(\tilde{x}\right)\right\}^{-1} = \int_0^\infty \left\{\left(\sum_{i=1}^{r} x_i^\beta\right) + (n-r)x_r^\beta + z^\beta\right\}^{-(r+1)} z^{\beta-1} dz$$

$$= \left(r\beta\right)^{-1}\left[\int_0^\infty r\beta\left\{\left(\sum_{i=1}^{r} x_i^\beta\right) + (n-r)x_r^\beta + z^\beta\right\}^{-(r+1)} z^{\beta-1} dz\right]$$

$$= -\left(r\beta\right)^{-1}\left[\left\{\left(\sum_{i=1}^{r} x_i^\beta\right) + (n-r)x_r^\beta + z^\beta\right\}^{-r}\right]_{z=0}^{z=\infty}$$

$$= \left(r\beta\right)^{-1}\left\{\left(\sum_{i=1}^{r} x_i^\beta\right) + (n-r)x_r^\beta\right\}^{-r} .$$

Therefore,

$$k(\tilde{x}) = r\beta\left\{\left(\sum_{i=1}^{r} x_i^\beta\right) + (n-r)x_r^\beta\right\}^{r} . \tag{4.5}$$

Substituting (4.5) in (4.4) we obtain,

$$\tilde{f}(z) = r\beta\left\{\left(\sum_{i=1}^{r} x_i^\beta\right) + (n-r)x_r^\beta\right\}^{r}\left\{\left(\sum_{i=1}^{r} x_i^\beta\right) + (n-r)x_r^\beta + z^\beta\right\}^{-(r+1)} z^{\beta-1} \; ; \quad z > 0 . \tag{4.6}$$

Letting $u = z^\beta$, equation (4.6) reduces to

$$\tilde{f}(u) = r\left\{\left(\sum_{i=1}^{r} x_i^{\beta}\right) + (n-r)x_r^{\beta}\right\}^{r} \left\{\left(\sum_{i=1}^{r} x_i^{\beta}\right) + (n-r)x_r^{\beta} + u\right\}^{-(r+1)} ;\ u > 0 . \quad (4.7)$$

Note that one can express the Pareto(α, κ) density in the form

$$f(x;\alpha,\kappa) = \kappa\alpha^{\kappa}(\alpha + x)^{-(\kappa+1)};\ \text{x} > 0, \quad (4.8)$$

where $\alpha > 0$, $\kappa > 0$ are parameters. Thus, the density $\tilde{f}$ given in equation (4.7) is a Pareto density with parameters $\kappa = r$ and $\alpha = \left\{\left(\sum_{i=1}^{r} x_i^{\beta}\right) + (n-r)x_r^{\beta}\right\}$. For the uncensored case, equation (4.7) reduces to

$$\tilde{f}(u) = n\left\{\sum_{i=1}^{n} x_i^{\beta}\right\}^{n} \left\{\left(\sum_{i=1}^{n} x_i^{\beta}\right) + u\right\}^{-(n+1)} ;\ u > 0 ,$$

which is a Pareto density with parameters $\kappa = n$ and $\alpha = \sum_{i=1}^{n} x_i^{\beta}$. If the probability density function of a Pareto distribution is expressed in the form given by equation (4.8) then the pth percentile of the distribution is given by

$$u_p = \alpha\left[(1-p)^{-1/\kappa} - 1\right].$$

Thus, if $U \sim Pareto(\alpha, \kappa)$, then

$$P\left[\alpha\left\{(1-p_1)^{-1/\kappa} - 1\right\} \le U \le \alpha\left\{(p_2)^{-1/\kappa} - 1\right\}\right] = 1 - (p_1 + p_2),$$

for $p_1,\ p_2\ \in (0,\ 1)$ and $p_1 + p_2 \in (0, 1)$. If one is to use the density function given in equation (4.7) as an approximate predictive density for Z^{β}, then an approximate $(1-p)100\%$ prediction interval about a future Z can be obtained as

$$P\left[\alpha^{1/\beta}\left\{(1-p_1)^{-1/r}-1\right\}^{1/\beta} \le Z \le \alpha^{1/\beta}\left\{(p_2)^{-1/r}-1\right\}^{1/\beta}\right] \approx 1-(p_1+p_2) \qquad (4.9)$$

with $p_1 + p_2 = p$ and $\alpha = \left\{\left(\sum_{i=1}^{r} x_i^{\beta}\right) + (n-r)x_r^{\beta}\right\}$.

At this point, we propose replacing the unknown shape parameter β in (4.7) and (4.9) by its bias- corrected maximum likelihood estimate based on $\tilde{x} = (x_1,\ x_2\ ,\ ...,\ x_r)$ or by the simple estimator for β proposed by Jayawradhana and Samaranayake (2003). In essence this estimator is very similar to the inverse of the simple linear estimate proposed by Engelhardt (1975) and Engelhardt and Bain (1977) for $1/\beta$. The only difference is that while the estimator proposed by the above authors is bias corrected for estimating $1/\beta$, that proposed by Jayawardhana and Samaranayake (2003) is bias corrected for β. This simple estimator is given by

$$\hat{\beta} = \frac{nk_{r,n}^{*}}{\sum_{i=1}^{r-1}\left(y_r - y_i\right)} \quad \text{for } r < n \qquad (4.10a)$$

and

$$\hat{\beta} = \frac{nk_{n}^{*}}{-\sum_{i=1}^{s} y_i + \left(\frac{s}{n-s}\right)\sum_{i=s+1}^{n} y_i} \quad \text{for } r = n, \qquad (4.10b)$$

where s = [0.84n] and $y_i = \ln x_i$. The Details of the bias correction procedures for both types of estimates and the unbiasing constants $k^*_{r,n}$ and k^*_n needed for this process are given in Jayawardhana and Samaranayake (2003).

Note that this "plug-in" approach deviates from the procedure suggested by Lejeune and Faulkenberry (1982) which requires β to be replaced by the closed form expression of its maximum likelihood estimate based on both $\tilde{x}$ and z. Since no closed form expression exists, the above "plug-in" approach was taken as a viable alternative. This substitution yields the predictive density

$$\hat{f}(z) = r\hat{\beta}\left\{\left(\sum_{i=1}^{r} x_i^{\hat{\beta}}\right) + (n-r)x_r^{\hat{\beta}}\right\}^r \left\{\left(\sum_{i=1}^{r} x_i^{\hat{\beta}}\right) + (n-r)x_r^{\hat{\beta}} + z^{\hat{\beta}}\right\}^{-(r+1)} z^{\hat{\beta}-1};\ z > 0, \qquad (4.11)$$

and the prediction interval

$$\left[\hat{\alpha}^{1/\hat{\beta}}\left\{(1-p_1)^{-1/r} - 1\right\}^{1/\hat{\beta}}, \hat{\alpha}^{1/\hat{\beta}}\left\{(p_2)^{-1/r} - 1\right\}^{1/\hat{\beta}}\right] \qquad (4.12)$$

for Z where $\hat{\alpha} = \left\{\left(\sum_{i=1}^{r} x_i^{\hat{\beta}}\right) + (n-r)x_r^{\hat{\beta}}\right\}$.

Using the parameterization of equation (4.8) to characterize Pareto densities, the mean and the variance of the random variable are $\alpha(\kappa-1)^{-1}$ and $\alpha^2\kappa(\kappa-2)^{-1}(\kappa-1)^{-2}$ respectively, provided $\kappa > 1$ and $\kappa > 2$. As discussed in Jayawardhana and Samaranayake (2003), the maximum likelihood predictive density method yields a density that has a larger variance than the variance of the underlying distribution of the future observations. For the Pareto distribution, an increase in variance necessarily increases the mean. This shift in the mean adversely affects the extreme percentiles and the proposed prediction bounds, as observed in simulation results. To rectify this problem, we came up with several A*d-hoc* modifications to the original bounds. These variations amount to replacing r in the expressions for the prediction bounds by $r-1$ and $r-2$ for upper bounds and $r+1$ and $r+2$ for the lower bounds.

5. PREDICTION BOUNDS FOR A WEIBULL RANDOM VARIABLE - TYPE I CENSORED CASE

Let X_1, X_2, ..., X_R be an ordered set of Type I censored past observations censored at time T from a set of n observations ($R \le n$) from a Weibull distribution with scale parameter θ and shape parameter β and let Z denote a future observation from the same Weibull distribution. Then the joint probability density of X_1, X_2, ..., X_R and Z is given by

$$f(x_1, x_2, \ldots, x_r, z; \theta, \beta) = (n!)\left\{(n-r)!\right\}^{-1} \beta^{r+1} \theta^{-(r+1)\beta} \left(\prod_{i=1}^{r} x_i^{\beta-1}\right) z^{\beta-1} \times$$

$$\exp\left[-\left\{\left(\sum_{i=1}^{r} x_i^{\beta}\right) + (n-r)T^{\beta} + z^{\beta}\right\}\theta^{-\beta}\right],$$

for $0 < x_1 < x_2$, ..., $< x_r$, $0 \le r \le n$. Using a procedure similar to that used in the Type II case, it can be shown that the maximum likelihood estimate of θ based on observations x_1, x_2, ..., x_r and z is given by

$$\hat{\theta} = \left[(r+1)^{-1} \left\{ \left(\sum_{i=1}^{r} x_i^{\hat{\beta}} \right) + (n-r)T^{\hat{\beta}} + z^{\hat{\beta}} \right\} \right]^{1/\hat{\beta}}$$

where $\hat{\beta}$ is the maximum likelihood estimate of β. Using the method proposed by Lejeune and Faulkenberry (1982) (similar to the Type II case) it can be shown that the predictive density function of Z given $R = r$, $X_i = x_i$, $i = 1, 2, ..., r$ is

$$\tilde{f}(z) = k(\tilde{x}) \left\{ \left(\sum_{i=1}^{r} x_i^{\beta} \right) + (n-r)T^{\beta} + z^{\beta} \right\}^{-(r+1)} z^{\beta-1};\ z > 0$$

under the assumption that β is known. Analogous to the Type II case we can also show that

$$k(\tilde{x}) = r\beta \left\{ \left(\sum_{i=1}^{r} x_i^{\beta} \right) + (n-r)T^{\beta} \right\}^{r};$$

$$\tilde{f}(z) = r\beta \left\{ \left(\sum_{i=1}^{r} x_i^{\beta} \right) + (n-r)T^{\beta} \right\}^{r} \left\{ \left(\sum_{i=1}^{r} x_i^{\beta} \right) + (n-r)T^{\beta} + z^{\beta} \right\}^{-(r+1)} z^{\beta-1}\ ;\ z > 0,$$

and letting $u = z^{\beta}$ as before, the predictive density of $U = Z^{\beta}$ is given by

$$\tilde{f}(u) = r \left\{ \left(\sum_{i=1}^{r} x_i^{\beta} \right) + (n-r)T^{\beta} \right\}^{r} \left\{ \left(\sum_{i=1}^{r} x_i^{\beta} \right) + (n-r)T^{\beta} + u \right\}^{-(r+1)} \quad ;\ u > 0\,. \tag{5.1}$$

Equation (5.1) has the form of a Pareto density with parameters $\kappa = r$ and $\alpha = \left\{ \left(\sum_{i=1}^{r} x_i^{\beta} \right) + (n-r)T^{\beta} \right\}$. If one is to use this as an approximate predictive density for Z^{β}, then an approximate $(1-p)100\%$ prediction interval about a future Z can be obtained by

$$P\left\{ \alpha^{1/\beta} \left[(1-p_1)^{-1/r} - 1 \right]^{1/\beta} \le Z \le \alpha^{1/\beta} \left[(p_2)^{-1/r} - 1 \right]^{1/\beta} \right\} \approx 1 - (p_1 + p_2) \qquad (5.2)$$

with $p_1 + p_2 = p$ and $\alpha = \left\{ \left(\sum_{i=1}^{r} x_i^{\beta} \right) + (n-r)T^{\beta} \right\}$.

As was done in the Type II censored case, we could replace the unknown shape parameter β by its maximum likelihood estimate or a simple estimate, based on x_1, x_2, ..., x_r and corrected for bias. While unbiasing constants for the MLE are given in Jayawardhana and Samaranayake (2003) for the Type II censored case, such constants are not available for MLEs based on Type I censored data. The reason for this is that while the bias is a function of *r/n* and *n* only for Type II censored data, with *r* predetermined and hence known, it is not the case for Type I censored situations. We addressed this problem by pretending that the observed data came from a Type II censoring scheme. In other words, once the observations are made at the fixed censoring time *T*, and the number of failures, *R*, by time *T* was observed to be equal to *r*, the unbiasing constant corresponding to the case *r/n* was selected from the table of values computed for the Type II case. If a simple estimator is to be used instead of the MLE, then there is the additional problem of not having such a simple estimator available for the Type I case. Instead of attempting to derive a simple estimator for the Type I case, we again pretended that the data arose from a Type II scheme, and used the simple estimator and unbiasing constants derived for that case. While this "seat-of-the-pants" approach is not theoretically justified, we attempted this in some initial test runs to determine whether such a naive solution would work. Since these initial runs gave promising results, this approach was retained. Complete simulation results show that the coverage probabilities obtained using this approach are, in most cases, comparable to those obtained for the Type II case.

Returning back to the derivation of the prediction bounds, this substitution of $\hat{\beta}$ for β in (5.1) and (5.2) yields the predictive density

$$\hat{f}(z) = r\hat{\beta}\left\{\left(\sum_{i=1}^{r} x_i^{\hat{\beta}}\right) + (n-r)T^{\hat{\beta}}\right\}^{r} \left\{\left(\sum_{i=1}^{r} x_i^{\hat{\beta}}\right) + (n-r)T^{\hat{\beta}} + z^{\hat{\beta}}\right\}^{-(r+1)} z^{\hat{\beta}-1};\ z > 0 \qquad (5.3)$$

and the prediction interval

$$\left[\hat{\alpha}^{1/\hat{\beta}}\left\{(1-p_1)^{-1/r} - 1\right\}^{1/\hat{\beta}}, \hat{\alpha}^{1/\hat{\beta}}\left\{(p_2)^{-1/r} - 1\right\}^{1/\hat{\beta}}\right] \qquad (5.4)$$

where $\hat{\alpha} = \left\{\left(\sum_{i=1}^{r} x_i^{\hat{\beta}}\right) + (n-r)T^{\hat{\beta}}\right\}$.

As was done in the Type II case, we considered variations by replacing r in the expressions for the prediction bounds by $r-1$ and $r-2$ for upper bounds and $r+1$ and $r+2$ for the lower bounds. For Type I censored case, additional modifications involving terms $r-3$ and $r+3$ were also used. In some Type I censored samples, we encountered situations in which $r-3<0$. In such cases we used $r-2$. Similar adjustments were made if $r-2<0$.

6. AN EXAMPLE

Consider the censored Weibull data (n = 10, r = 5) given by Lawless (1973): 50.5, 71.3, 84.6, 98.7 and 103.8. Based on these data, we found the maximum likelihood estimate of β to be 4.199. The unbiased estimate of β is (0.638)(4.199) which is equal to 2.678962 (see Table 2 in Jayawardhana and Samaranayake (2003) for unbiasing constants).

(i) A 90% lower prediction limit for a future observation is

$$\left[\left\{\left(\sum_{i=1}^{r} x_i^{\hat{\beta}}\right) + (n-r)x_r^{\hat{\beta}}\right\}\left\{(1-p)^{-1/r} - 1\right\}\right]^{1/\hat{\beta}}$$

$$= \left[\left\{746439.83 + (10-5)103.8^{2.678962}\right\}\left\{0.9^{-1/5} - 1\right\}\right]^{1/2.678962} = 53.52.$$

Engelhardt and Bain (1982) gave the same example and found a 90% lower prediction limit for a future observation to be 59.1 using their method. Fertig *et al.* (1980) found a 90% lower prediction limit for a future observation to be 56.98. Mee and Kushary (1994) found a 90% lower prediction limit to be 56.6. A 90% lower prediction limit using our method is more conservative than the above three results.

(ii) Fertig *et al.* (1980) found a 90% upper prediction limit for a future observation to be 185.15. Our method results in

$$\left[\left\{746439.83 + (10-5)103.8^{2.678962}\right\}\left\{0.1^{-1/5} - 1\right\}\right]^{1/2.678962} = 184.33$$

for a 90% upper prediction limit. Our 90% upper prediction limit is slightly lower.

7. MONTE CARLO SIMULATION

All simulation work was done on an HP 9000/735 machine with HP-UX operating system in double precision and the simulation programs were written in FORTRAN. These programs called International Mathematical and Statistical Libraries (IMSL) subroutines, where applicable. They were executed using a FORTRAN 77 compiler. Each simulation was started by setting the seed to 123457.

For Type II censored case, using θ = 1 and β = 1, samples of Weibull random variables (in essence these are exponential random variables) of sizes n = 10, 20, 50, and 100 were generated using the IMSL subroutines DRNWIB. Several intermediate sample sizes between 20 and 50 were also run later, after initial simulations showed the unmodified method producing reasonable 95% and 99% bounds for sample sizes 50 and greater, but not for n = 20. These additional simulations show that 95% and 99% bounds with close to nominal coverage probabilities result when sample size is 25 or more. Thus, the case n = 25 is also included. Differing values of θ were not considered because prediction bounds are invariant with respect to θ (i.e. to changes in scale). Preliminary simulations were carried out by setting $\beta = 1$, 2, and 5 prior to our realization that the coverage probabilities obtained via the simulation are invariant with respect to β. Proof of this claim is given in the appendix.

Observations in each sample were arranged in increasing order using the IMSL subroutine DSVRGN so as to facilitate the censoring. For Type II censored case, censoring schemes used were (i) n = 10 and r = 3, 5, 10; (ii) n = 20 and r = 10, 15, 20; (iii) n = 25 and r = 15, 20, 25; (iv) n = 50 and r = 30, 40, 50; and (v) n = 100 and r = 50, 75, 100.

For Type I censored case, we again used $\theta = 1$ and $\beta =$ 1. In the Type II censoring situations, the ratio *r/n* is a clear-cut and a commonly used measure of the degree of censoring. Authors such as Engelhardt and Bain (1982) have used various combinations of *r* and *n* to study the behavior of their proposed prediction bounds. Lacking such a direct indicator for the Type I case, we selected censoring times *T* such that $P(X > T) = \gamma$ for γ = 0.5, 0.25, 0.10, and 0.05. It can be easily shown that $P(X > T) = E[(n - R)/n]$, where *R* is the random variable representing the number of test items that would fail by time *T*. Letting *c* = $100 \times E[(n - R)/n]$ denote the expected percentage of censored sample units, the above censoring scheme produces *c* values equal to 50%, 25%, 10%, and 5%. While pre-specifying censoring times *T* in this fashion is unfeasible in actual life testing situation, since this requires the knowledge of the unknown parameters, we adopted this approach in order to investigate how various levels of censoring affects the coverage probabilities of the proposed bounds, just like looking at various combinations of *r* and *n* to determine the coverage properties of the bounds under Type II censoring. In other words, the imposition of this censoring scheme is not intended as an example of how censoring times may be determined in practice, but merely as a tool for investigating the coverage properties of the prediction bounds under various degrees of censoring.

Using the IMSL subroutine DBCONG, we maximized the log likelihood function for each parameter combination and found the maximum likelihood estimate of β. These estimates were bias corrected by using the unbiasing constants given in Table 2 of Jayawardhana and Samaranayake (2003). Simple estimates were made unbiased by using the unbiasing constants given in Table 1 of the same publication. In all cases the 1st, 2.5th, 5th, 10th, 90th, 95th, 97.5th, and 99th percentile points of the underlying distribution were estimated using expressions (4.12) for Type II censored case and equation (5.4) for Type I censored case.

The estimation of percentile points was repeated with $\kappa = r$ replaced by $r+1$, $r+2$ for lower percentiles and $\kappa = r$ replaced by $r-1$ and $r-2$ for upper percentiles. Additional runs with *r* replaced by *r*+3 for lower percentiles and $r-3$ for upper percentiles were carried out for the Type I censoring schemes. As mentioned in Section 5, the adjustments $r-1$, $r-2$, and $r-3$

for the Type I case may not be feasible in situations where the number of failures $R = r$ is so small that these adjusted values become negative. In such cases we used the next best adjustment that yielded usable values. Thus, failing to use $r-3$ which is negative, we used $r-2$. If $r-2$ is also negative we used $r-1$, or r if $r-1$ was also unusable.

Using the theoretical distribution, the probability of a Weibull random variable being greater than the estimates of the lower percentile points, and the probability of a Weibull random variable being smaller than the estimates of the upper percentile points were calculated. This procedure was repeated 10,000 times and the corresponding probabilities were averaged for each combination of values of n and r for Type II censored case and n and c for Type I censored case.

Table 1 provides the results of the simulation for the Type II censored case using maximum likelihood estimates of β. Table 2 provides the results of the simulation for the Type II censored case using simple linear estimates of β. The first two columns of these tables give n and r. To conserve space only the coverage probabilities for the 90%, 95%, and 99% bounds are reported. Results for the other cases are available from the first author upon request.

Let X be a random variable from the theoretical Weibull distribution with parameters θ and β. For the Type II case, columns four, five, and six give estimates of E[P{X $\geq$ estimated 10^{th} percentile}], E[P{X $\geq$ estimated 5^{th} percentile}], and E[P{X $\geq$ estimated 1^{st} percentile}], respectively. Columns eight, nine, and ten give estimates of E[P{X $\leq$ estimated 90^{th} percentile}], E[P{X $\leq$ estimated 95^{th} percentile}], and E[P{X $\leq$ estimated 99^{th} percentile}], respectively.

Tables 3, 4, and 5 provide the results for the Type I censored case, for bounds based on the MLE, for sample sizes 10, 20, and 50, respectively. The first column of Tables 3, 4, and 5 gives c. Let X be a random variable from the corresponding Weibull distribution. Columns three, four, and five give estimates of E[P{X ≥ estimated 10^{th} percentile}], E[P{X ≥ estimated 5^{th} percentile}], and E[P{X ≥ estimated 1^{st} percentile}], respectively. Columns seven, eight, and nine give estimates of E[P{X ≤ estimated 90^{th} percentile}], E[P{X ≤ estimated 95^{th} percentile}], and E[P{X ≤ estimated 99^{th} percentile}], respectively. Tables 6, 7, and 8 provide simulations results for the Type I case, for bounds based on the simple estimator. The format of these tables are the same as that for Tables 3, 4, and 5.

8. DISCUSSION OF RESULTS

For Type II censored data, simulation results for prediction bounds, using either the maximum likelihood or the simple estimates of β, show that our approximation method is adequate for constructing 90% lower and upper prediction limits. The only exception is for upper prediction limits based on small samples with heavy censoring (n = 10, r = 3), with the MLE used for $\hat{\beta}$. The proposed method also works reasonably well in yielding 95% and 99% upper and lower prediction bounds for moderate to large sample sizes ($n \geq 25$) without heavy censoring. For small samples and cases with heavy censoring, our proposed A*d-hoc* adjustment of replacing $\kappa = r$ by $r+1$ for lower prediction limits and by $r-1$ for upper prediction limits appears to work well for 95% prediction bounds but is slightly liberal for 99% prediction limits. Replacing $\kappa = r$ by $r+2$ for lower prediction limits and by $r-2$ for upper prediction limits appears to work for 99% prediction bounds except when there is heavy censoring. Even in cases with heavy censoring, the drop in coverage probability for 99% bounds is slight, noticeable only at the 3^{rd} decimal place. There is no marked difference in coverage probabilities between the 90% and 95% and bounds that use the MLE and the simple estimator of β. When constructing 99% upper prediction bounds, the MLE based prediction limits tend to give better coverages, especially with small samples. We recommend using the MLE rather than the simple estimator of β for 99% upper bounds in cases involving small to moderate samples. Table 9 gives our recommended methods for obtaining 90%, 95%, and 99% bounds under various sample sizes and censoring levels. These recommendations will provide bounds that give coverage probabilities very close to the nominal values, with very few cases yielding slightly liberal coverages.

For Type I censoring, we get different results. Using the maximum likelihood estimate of β, our method appears to work reasonably for constructing 90% , and 95% lower and upper prediction limits. Our proposed A*d-hoc* adjustments improve the coverage slightly even though it is not necessary to do so for moderate to large samples. Modifications are needed, however, for 99% intervals. For sample size 10, the adjustment $r+3$ for lower 99% bounds and $r-1$ for 99% upper prediction bounds are needed. For sample sizes 20 and 50, the adjustment r-1 is needed for upper 99% bounds . The adjustmeny $r+1$ is required for the lower 99% bounds when n=20. One feature of the results is that coverage probabilities tend to increase with heavier censoring.

When the simple estimate of β was used instead of MLE's, the coverage probabilities drop for some parameter combinations, especially when n = 10. Prediction bound based on the simple estimator also show a tendency to provide higher coverage probabilities under heavier censoring. While bounds based on the MLE's of β provide very reasonable coverage for Type I censored data, we recommend using limits based on simple estimates of β because of their ease of computation. Table 10 gives our recommendations on the appropriate method to use for various sample sizes and confidence levels. These recommendations will result in prediction bounds that provide coverage probabilities very close to nominal values under most of the parameter combinations considered in our simulation study. In the few cases where coverage is liberal, the shortfall is minimal. If it is essential that coverage probabilities stay at or above the nominal value for Type I censored situations, we recommend using bounds based on the MLE of β for situations involving small sample sizes.

9. CONCLUSIONS

A method for constructing approximate upper and lower prediction intervals for a future Weibull observation is proposed. Advantages of this method are (1) the bounds are easily computed, (2) the predictive density is well-known, (3) the percentiles are available in explicit form, (4) both upper and lower prediction limits can be obtained, and (5) the procedure works for both Type I and Type II censoring. Simulation results show that the coverage probabilities are slightly liberal in some situations. In many cases however, the coverage probabilities differ from the nominal levels only at the third decimal place. Ad hoc adjustments are proposed to address situations where coverage falls below nominal level. These adjustments provide a good remedy in almost all such situations.

Table 1: Type II Censoring
Coverage Probabilities for the Unmodified and Modified Methods Using Maximum Likelihood Estimates of β

			$P(Z \geq \hat{z}_p)$				$P(Z \leq \hat{z}_{1-p})$		
n	r	κ	p =0.10	p =0.05	p =0.01	κ	p =0.10	p =0.05	p =0.01
10	3	r	0.926	0.961	0.987	r	0.892	0.918	0.952
		$r+1$	0.944	0.969	0.989	$r-1$	0.931	0.951	0.976
		$r+2$	0.953	0.973	0.991	$r-2$	0.976	0.987	0.998
10	5	r	0.913	0.952	0.984	r	0.902	0.933	0.965
		$r+1$	0.926	0.958	0.986	$r-1$	0.929	0.952	0.975
		$r+2$	0.935	0.962	0.987	$r-2$	0.953	0.969	0.985
10	10	r	0.901	0.946	0.985	r	0.901	0.949	0.985
		$r+1$	0.908	0.950	0.986	$r-1$	0.923	0.962	0.989
		$r+2$	0.915	0.953	0.987	$r-2$	0.943	0.973	0.993
20	10	r	0.903	0.948	0.986	r	0.896	0.936	0.972
		$r+1$	0.911	0.952	0.987	$r-1$	0.914	0.947	0.978
		$r+2$	0.917	0.955	0.988	$r-2$	0.931	0.958	0.982
20	15	r	0.901	0.948	0.987	r	0.903	0.948	0.984
		$r+1$	0.907	0.951	0.988	$r-1$	0.917	0.956	0.986
		$r+2$	0.912	0.953	0.988	$r-2$	0.930	0.964	0.990
20	20	r	0.900	0.948	0.988	r	0.901	0.950	0.988
		$r+1$	0.904	0.950	0.988	$r-1$	0.912	0.957	0.990
		$r+2$	0.908	0.952	0.989	$r-2$	0.923	0.963	0.992
25	15	r	0.902	0.949	0.987	r	0.902	0.945	0.981
		$r+1$	0.908	0.952	0.988	$r-1$	0.915	0.953	0.984
		$r+2$	0.913	0.954	0.989	$r-2$	0.928	0.961	0.987
25	20	r	0.901	0.948	0.987	r	0.902	0.949	0.986
		$r+1$	0.905	0.951	0.988	$r-1$	0.913	0.956	0.988
		$r+2$	0.909	0.953	0.989	$r-2$	0.924	0.962	0.990
25	25	r	0.900	0.948	0.988	r	0.901	0.950	0.989
		$r+1$	0.904	0.950	0.989	$r-1$	0.910	0.956	0.990
		$r+2$	0.907	0.952	0.989	$r-2$	0.919	0.961	0.992

Table 1 continued: Type II Censoring
Coverage Probabilities for the Unmodified and Modified Methods Using Maximum Likelihood Estimates of β

			$P\left(Z \geq \hat{z}_p\right)$				$P\left(Z \leq \hat{z}_{1-p}\right)$		
n	r	κ	p =0.10	p =0.05	p =0.01	κ	p =0.10	p =0.05	p =0.01
50	30	r	0.901	0.949	0.989	r	0.900	0.947	0.985
		$r+1$	0.904	0.951	0.989	$r-1$	0.908	0.951	0.986
		$r+2$	0.907	0.952	0.989	$r-2$	0.915	0.956	0.988
50	40	r	0.901	0.949	0.989	r	0.901	0.949	0.988
		$r+1$	0.903	0.950	0.989	$r-1$	0.907	0.953	0.989
		$r+2$	0.905	0.952	0.989	$r-2$	0.912	0.956	0.990
50	50	r	0.900	0.949	0.989	r	0.900	0.950	0.989
		$r+1$	0.902	0.950	0.989	$r-1$	0.905	0.953	0.990
		$r+2$	0.904	0.951	0.990	$r-2$	0.909	0.956	0.991
100	50	r	0.901	0.950	0.989	r	0.900	0.947	0.986
		$r+1$	0.903	0.951	0.989	$r-1$	0.904	0.950	0.987
		$r+2$	0.904	0.952	0.990	$r-2$	0.909	0.953	0.988
100	75	r	0.900	0.950	0.989	r	0.900	0.949	0.989
		$r+1$	0.901	0.950	0.990	$r-1$	0.903	0.951	0.989
		$r+2$	0.903	0.951	0.990	$r-2$	0.906	0.953	0.990
100	100	r	0.900	0.950	0.990	r	0.900	0.950	0.990
		$r+1$	0.901	0.950	0.990	$r-1$	0.902	0.951	0.990
		$r+2$	0.902	0.951	0.990	$r-2$	0.905	0.953	0.990

Table 2: Type II Censoring
Coverage Probabilities for the Unmodified and Modified Methods Using Simple Estimates of β

			$P\left(Z \geq \hat{z}_p\right)$				$P\left(Z \leq \hat{z}_{1-p}\right)$		
n	r	κ	p =0.10	p =0.05	p =0.01	κ	p =0.10	p =0.05	p =0.01
10	3	r	0.933	0.964	0.986	r	0.901	0.921	0.946
		$r+1$	0.949	0.970	0.988	$r-1$	0.930	0.945	0.963
		$r+2$	0.957	0.974	0.989	$r-2$	0.962	0.971	0.982
10	5	r	0.913	0.951	0.984	r	0.901	0.933	0.964
		$r+1$	0.925	0.957	0.985	$r-1$	0.929	0.951	0.974
		$r+2$	0.934	0.962	0.987	$r-2$	0.953	0.968	0.983
10	10	r	0.900	0.946	0.985	r	0.902	0.949	0.985
		$r+1$	0.908	0.950	0.986	$r-1$	0.923	0.962	0.989
		$r+2$	0.915	0.953	0.987	$r-2$	0.943	0.972	0.992
20	10	r	0.905	0.950	0.987	r	0.900	0.938	0.974
		$r+1$	0.913	0.954	0.988	$r-1$	0.917	0.950	0.979
		$r+2$	0.920	0.957	0.988	$r-2$	0.934	0.960	0.983
20	15	r	0.901	0.947	0.987	r	0.902	0.947	0.984
		$r+1$	0.906	0.950	0.988	$r-1$	0.916	0.956	0.987
		$r+2$	0.911	0.953	0.988	$r-2$	0.929	0.964	0.989
20	20	r	0.900	0.947	0.988	r	0.901	0.950	0.988
		$r+1$	0.904	0.950	0.988	$r-1$	0.912	0.957	0.990
		$r+2$	0.908	0.952	0.989	$r-2$	0.923	0.963	0.992
25	15	r	0.903	0.950	0.988	r	0.903	0.946	0.981
		$r+1$	0.909	0.952	0.988	$r-1$	0.916	0.953	0.984
		$r+2$	0.914	0.955	0.989	$r-2$	0.928	0.961	0.987
25	20	r	0.902	0.949	0.988	r	0.903	0.949	0.986
		$r+1$	0.906	0.951	0.988	$r-1$	0.913	0.956	0.988
		$r+2$	0.910	0.953	0.989	$r-2$	0.924	0.962	0.990
25	25	r	0.900	0.948	0.988	r	0.901	0.950	0.988
		$r+1$	0.903	0.950	0.988	$r-1$	0.910	0.956	0.990
		$r+2$	0.906	0.951	0.989	$r-2$	0.919	0.961	0.992

Table 2 continued: Type II Censoring
Coverage Probabilities for the Unmodified and Modified Methods Using Simple Estimates of β

			$P\left(Z \geq \hat{z}_p\right)$				$P\left(Z \leq \hat{z}_{1-p}\right)$		
n	r	κ	p =0.10	p =0.05	p =0.01	κ	p =0.10	p =0.05	p =0.01
50	30	r	0.901	0.949	0.989	r	0.901	0.947	0.985
		$r+1$	0.904	0.951	0.989	$r-1$	0.908	0.951	0.986
		$r+2$	0.907	0.952	0.989	$r-2$	0.915	0.956	0.988
50	40	r	0.900	0.949	0.989	r	0.901	0.949	0.988
		$r+1$	0.903	0.950	0.989	$r-1$	0.906	0.953	0.989
		$r+2$	0.905	0.951	0.989	$r-2$	0.912	0.956	0.990
50	50	r	0.900	0.949	0.989	r	0.900	0.950	0.989
		$r+1$	0.902	0.950	0.989	$r-1$	0.905	0.953	0.990
		$r+2$	0.903	0.951	0.989	$r-2$	0.909	0.956	0.991
100	50	r	0.901	0.950	0.989	r	0.900	0.947	0.986
		$r+1$	0.903	0.951	0.989	$r-1$	0.904	0.950	0.987
		$r+2$	0.904	0.952	0.990	$r-2$	0.909	0.953	0.988
100	75	r	0.899	0.949	0.989	r	0.899	0.948	0.988
		$r+1$	0.900	0.949	0.989	$r-1$	0.902	0.950	0.989
		$r+2$	0.901	0.950	0.989	$r-2$	0.905	0.952	0.989
100	100	r	0.900	0.949	0.990	r	0.900	0.950	0.990
		$r+1$	0.901	0.950	0.990	$r-1$	0.902	0.951	0.990
		$r+2$	0.902	0.950	0.990	$r-2$	0.905	0.953	0.990

Table 3: Type I Censoring
Coverage Probabilities for the Unmodified and Modified Methods Using Maximum Likelihood Estimates of β

c = proportion of censoring and n = 10

		$P(Z \geq \hat{z}_p)$				$P(Z \leq \hat{z}_{1-p})$		
c	κ	p =0.10	p =0.05	p =0.01	κ	p =0.10	p =0.05	p =0.01
50%	r	0.929	0.968	0.991	r	0.948	0.970	0.989
	$r+1$	0.942	0.973	0.992	$r-1$	0.964	0.980	0.992
	$r+2$	0.949	0.976	0.993	$r-2$	0.977	0.987	0.995
	$r+3$	0.954	0.978	0.993	$r-3$	0.986	0.992	0.997
25%	r	0.924	0.961	0.990	r	0.928	0.961	0.988
	$r+1$	0.933	0.965	0.991	$r-1$	0.945	0.971	0.991
	$r+2$	0.940	0.969	0.992	$r-2$	0.961	0.980	0.994
	$r+3$	0.945	0.971	0.992	$r-3$	0.974	0.987	0.996
10%	r	0.911	0.952	0.987	r	0.913	0.954	0.986
	$r+1$	0.919	0.957	0.988	$r-1$	0.933	0.965	0.990
	$r+2$	0.926	0.960	0.989	$r-2$	0.950	0.975	0.993
	$r+3$	0.931	0.963	0.990	$r-3$	0.965	0.983	0.995
5%	r	0.906	0.949	0.986	r	0.908	0.951	0.985
	$r+1$	0.914	0.953	0.987	$r-1$	0.928	0.963	0.989
	$r+2$	0.921	0.957	0.988	$r-2$	0.947	0.974	0.993
	$r+3$	0.926	0.960	0.989	$r-3$	0.962	0.982	0.995

Table 4: Type I Censoring
Coverage Probabilities for the Unmodified and Modified Methods Using Maximum Likelihood Estimates of β
c = proportion of censoring and n = 20

		$P\left(Z \geq \hat{z}_p\right)$				$P\left(Z \leq \hat{z}_{1-p}\right)$		
c	κ	p =0.10	p =0.05	p =0.01	κ	p =0.10	p =0.05	p =0.01
50%	r	0.912	0.957	0.990	r	0.926	0.959	0.986
	$r+1$	0.921	0.961	0.991	$r-1$	0.939	0.967	0.989
	$r+2$	0.928	0.964	0.991	$r-2$	0.952	0.974	0.992
	$r+3$	0.934	0.967	0.992	$r-3$	0.963	0.980	0.994
25%	r	0.907	0.952	0.990	r	0.916	0.957	0.989
	$r+1$	0.913	0.955	0.990	$r-1$	0.928	0.964	0.991
	$r+2$	0.918	0.958	0.990	$r-2$	0.940	0.971	0.993
	$r+3$	0.922	0.960	0.991	$r-3$	0.950	0.976	0.994
10%	r	0.904	0.950	0.990	r	0.909	0.954	0.989
	$r+1$	0.909	0.953	0.990	$r-1$	0.920	0.961	0.991
	$r+2$	0.913	0.955	0.990	$r-2$	0.931	0.967	0.993
	$r+3$	0.917	0.957	0.990	$r-3$	0.942	0.973	0.994
5%	r	0.903	0.949	0.988	r	0.905	0.952	0.988
	$r+1$	0.907	0.952	0.989	$r-1$	0.917	0.959	0.990
	$r+2$	0.911	0.954	0.989	$r-2$	0.928	0.966	0.992
	$r+3$	0.915	0.956	0.990	$r-3$	0.938	0.971	0.994

Table 5: Type I Censoring
Coverage Probabilities for the Unmodified and Modified Methods Using Maximum Likelihood Estimates of β
c = proportion of censoring and n = 50

		$P(Z \geq \hat{z}_p)$				$P(Z \leq \hat{z}_{1-p})$		
c	κ	p =0.10	p =0.05	p =0.01	κ	p =0.10	p =0.05	p =0.01
50%	r	0.904	0.952	0.990	r	0.914	0.954	0.987
	$r+1$	0.908	0.954	0.990	$r-1$	0.921	0.959	0.989
	$r+2$	0.912	0.956	0.990	$r-2$	0.928	0.963	0.990
	$r+3$	0.915	0.957	0.991	$r-3$	0.935	0.967	0.991
25%	r	0.903	0.951	0.990	r	0.907	0.954	0.989
	$r+1$	0.905	0.952	0.990	$r-1$	0.913	0.957	0.990
	$r+2$	0.908	0.953	0.990	$r-2$	0.919	0.960	0.991
	$r+3$	0.910	0.954	0.990	$r-3$	0.924	0.964	0.992
10%	r	0.902	0.950	0.989	r	0.904	0.952	0.990
	$r+1$	0.904	0.951	0.990	$r-1$	0.909	0.955	0.990
	$r+2$	0.906	0.952	0.990	$r-2$	0.914	0.958	0.991
	$r+3$	0.908	0.953	0.990	$r-3$	0.918	0.961	0.992
5%	r	0.901	0.950	0.989	r	0.902	0.951	0.990
	$r+1$	0.903	0.951	0.990	$r-1$	0.907	0.954	0.990
	$r+2$	0.905	0.952	0.990	$r-2$	0.912	0.957	0.991
	$r+3$	0.907	0.953	0.990	$r-3$	0.917	0.960	0.992

Table 6: Type I Censoring
Coverage Probabilities for the Unmodified and Modified Methods Using Simple Estimates of β

c = proportion of censoring and n = 10

		$P\left(Z \geq \hat{z}_p\right)$				$P\left(Z \leq \hat{z}_{1-p}\right)$		
c	κ	p =0.10	p =0.05	p =0.01	κ	p =0.10	p =0.05	p =0.01
50%	r	0.897	0.939	0.974	r	0.901	0.934	0.967
	$r+1$	0.913	0.946	0.976	$r-1$	0.926	0.951	0.975
	$r+2$	0.922	0.951	0.977	$r-2$	0.948	0.966	0.983
	$r+3$	0.927	0.953	0.978	$r-3$	0.965	0.977	0.988
25%	r	0.900	0.937	0.980	r	0.905	0.945	0.980
	$r+1$	0.901	0.943	0.981	$r-1$	0.927	0.959	0.986
	$r+2$	0.909	0.947	0.983	$r-2$	0.946	0.971	0.990
	$r+3$	0.916	0.951	0.984	$r-3$	0.963	0.980	0.994
10%	r	0.889	0.937	0.981	r	0.903	0.947	0.983
	$r+1$	0.898	0.942	0.983	$r-1$	0.924	0.960	0.988
	$r+2$	0.906	0.946	0.984	$r-2$	0.944	0.971	0.992
	$r+3$	0.912	0.950	0.985	$r-3$	0.960	0.980	0.995
5%	r	0.891	0.939	0.982	r	0.902	0.948	0.984
	$r+1$	0.900	0.944	0.983	$r-1$	0.924	0.961	0.989
	$r+2$	0.907	0.948	0.985	$r-2$	0.943	0.972	0.992
	$r+3$	0.913	0.951	0.985	$r-3$	0.960	0.981	0.995

Table 7: Type I Censoring
Coverage Probabilities for the Unmodified and Modified Methods using Simple Estimates of β

c = proportion of censoring and n = 20

		$P\left(Z \geq \hat{z}_p\right)$				$P\left(Z \leq \hat{z}_{1-p}\right)$		
c	κ	p =0.10	p =0.05	p =0.01	κ	p =0.10	p =0.05	p =0.01
50%	r	0.895	0.944	0.984	r	0.904	0.942	0.977
	$r+1$	0.905	0.948	0.985	$r-1$	0.920	0.952	0.982
	$r+2$	0.912	0.952	0.986	$r-2$	0.934	0.962	0.986
	$r+3$	0.919	0.955	0.987	$r-3$	0.948	0.970	0.989
25%	r	0.892	0.942	0.985	r	0.904	0.947	0.984
	$r+1$	0.898	0.945	0.986	$r-1$	0.916	0.955	0.987
	$r+2$	0.903	0.948	0.986	$r-2$	0.929	0.963	0.990
	$r+3$	0.908	0.950	0.987	$r-3$	0.940	0.970	0.992
10%	r	0.896	0.945	0.987	r	0.906	0.952	0.988
	$r+1$	0.901	0.948	0.987	$r-1$	0.917	0.959	0.990
	$r+2$	0.906	0.950	0.988	$r-2$	0.929	0.966	0.992
	$r+3$	0.910	0.952	0.988	$r-3$	0.939	0.972	0.994
5%	r	0.901	0.948	0.988	r	0.906	0.953	0.988
	$r+1$	0.906	0.951	0.988	$r-1$	0.918	0.960	0.990
	$r+2$	0.910	0.953	0.989	$r-2$	0.929	0.966	0.992
	$r+3$	0.914	0.955	0.989	$r-3$	0.939	0.972	0.994

Table 8: Type I Censoring
Coverage Probabilities for the Unmodified and Modified Methods using Simple Estimates of β

c = proportion of censoring and n = 50

		$P\left(Z \geq \hat{z}_p\right)$				$P\left(Z \leq \hat{z}_{1-p}\right)$		
c	κ	p =0.10	p =0.05	p =0.01	κ	p =0.10	p =0.05	p =0.01
50%	r	0.897	0.947	0.988	r	0.902	0.946	0.983
	$r+1$	0.901	0.949	0.988	$r-1$	0.910	0.951	0.985
	$r+2$	0.905	0.951	0.989	$r-2$	0.917	0.955	0.987
	$r+3$	0.908	0.952	0.989	$r-3$	0.925	0.960	0.988
25%	r	0.896	0.946	0.988	r	0.901	0.948	0.987
	$r+1$	0.898	0.947	0.988	$r-1$	0.907	0.952	0.988
	$r+2$	0.900	0.948	0.988	$r-2$	0.912	0.955	0.989
	$r+3$	0.903	0.950	0.989	$r-3$	0.918	0.959	0.990
10%	r	0.900	0.949	0.989	r	0.904	0.953	0.990
	$r+1$	0.902	0.950	0.989	$r-1$	0.909	0.956	0.991
	$r+2$	0.904	0.951	0.990	$r-2$	0.914	0.959	0.992
	$r+3$	0.906	0.952	0.990	$r-3$	0.919	0.962	0.993
5%	r	0.914	0.958	0.992	r	0.911	0.959	0.993
	$r+1$	0.916	0.959	0.992	$r-1$	0.915	0.962	0.994
	$r+2$	0.917	0.960	0.992	$r-2$	0.920	0.965	0.994
	$r+3$	0.919	0.960	0.992	$r-3$	0.925	0.967	0.995

Table 9: Recommended Methods for Type II Censored Data

Unmodified method denoted by *r*, other methods denoted by modifications *r*-1, *r*-2, *r*+1, and *r*+2.

Use of simple estimate of β assumed unless otherwise noted (as MLE)

		$P(Z \geq \hat{z}_p)$			$P(Z \leq \hat{z}_{1-p})$		
n	*r*	*p* = 0.10	*p* = 0.05	*p* = 0.01	*p* = 0.10	*p* = 0.05	*p* = 0.01
10	3	*r*	*r*	*r* +2	*r*	*r* -2	*r*-2, MLE
10	5	*r*	*r*	*r* +2	*r*	*r* -1	*r*-2, MLE
10	10	*r*	*r* +1	*r* +2	*r*	*r* -1	*r* -2
20	10	*r*	*r*	*r* +2	*r*	*r* -1	*r* -2
20	15	*r*	*r* +1	*r* +2	*r*	*r* -1	*r* -2
20	20	*r*	*r* +1	*r* +2	*r*	*r* -1	*r* -2
25	15	*r*	*r*	*r* +2	*r*	*r* -1	*r* -2
25	20	*r*	*r* +1	*r* +2	*r*	*r* -1	*r* -2
25	25	*r*	*r* +1	*r* +2	*r*	*r* -1	*r* -2
50	30	*r*	*r*	*r* +2	*r*	*r* -1	*r* -2
50	40	*r*	*r*	*r* +2	*r*	*r* -1	*r* -2
50	50	*r*	*r*	*r* +2	*r*	*r* -1	*r* -2
100	50	*r*	*r*	*r* +2	*r*	*r* -1	*r* -2
100	75	*r*	*r*	*r* +2	*r*	*r* -1	*r* -2
100	100	*r*	*r*	*r* +2	*r*	*r* -1	*r* -2

Table 10: Recommended Bounds for Type I Censored Data

Unmodified method denoted by *r*, other methods denoted by modifications *r*-1, *r*-2, *r*-3, *r*+1, *r*+2, and *r*+3.

Use of simple estimate of β assumed unless otherwise noted (as MLE)

	$P(Z \geq \hat{z}_p)$			$P(Z \leq \hat{z}_{1-p})$		
n	*p* = 0.10	*p* = 0.05	*p* = 0.01	*p* = 0.10	*p* = 0.05	*p* = 0.01
10	*r* +1	*r* +3	*r*+3 MLE	*r*	*r* -1	*r* -3*
20	*r* +1	*r* +2	*r*+1 MLE	*r*	*r* -1	*r* -3
50	*r* +1	*r* +2	*r* +3	*r*	*r* -1	*r* -3

* If heavy censoring, use *r* -1 and MLE

References

Antle, C. E. and Rademaker, F. (1972). "An upper confidence limit on the maximum of m future observations from a Type 1 extreme-value distribution", *Biometrika*, 59, 475-477.

Engelhardt, M. (1975). "On Simple Estimation of Parameters of the Weibull or Extreme-Value Distribution", *Technometrics*, 17, 369-374.

Engelhardt, M. and Bain, L. J. (1977). "Simplified Statistical Procedures for the Weibull or Extreme-Value Distribution", *Technometrics*, 19, 323-331.

Engelhardt, M. and Bain, L. J. (1979). "Prediction Limits and Two-Sample Problems with Complete or Censored Weibull Data", *Technometrics*, 21, 233-237.

Engelhardt, M. and Bain, L. J. (1982). "On Prediction Limits for Samples from a Weibull or Extreme-Value Distribution", *Technometrics*, 24, 147-150.

Fertig, K. W., Mayer, M. and Mann, N. R. (1980). "On Constructing Prediction Intervals for Samples from a Weibull or Extreme Value Distribution", *Technometrics*, 22, 567-573.

Jayawardhana, A. A. and Samaranayake, V. A. (2003). "Prediction Bounds in Accelerated Life Testing: Weibull Models with Inverse Power Relationship", *J. Quality Technology*, 35, 89-103.

Lawless, J. F. (1973). "On the Estimation of Safe Life when the Underlying Life Distribution is Weibull", *Technometrics*, 15, 857-865.

Lejeune, M. and Faulkenberry, G. D. (1982). "A Simple Predictive Function", *Journal of the American Statistical Association*, 77, 654-657.

Mann, N.R. (1968). "Results on Statistical Estimation and Hypothesis Testing with Application to Weibull and Extreme-Value Distributions", *Aerospace Research Laboratories Report* ARL 68-0068, Office of aerospace Research, U.S. Air Force, Write-Patterson Air Force base, Ohio.

Mann, N. R. (1970). "Warranty Periods Based on Three Ordered Sample Observations from a Weibull Population", *IEEE Transactions on Reliability*, R-19, 167-171.

Mann, N. R. (1976). "Warranty Periods for Production Lots Based on Fatigue-Test Data", *Engineering Fracture Mechanics*, 8, 123-130.

Mann, N. R. and Saunders, S. C. (1969). "On Evaluation of Warranty Assurance when Life has a Weibull Distribution", *Biometrika*, 56, 615-625.

Mann, N. R. Schafer, R. E. and Singpurwalla, N. D., (1974). *Methods for Statistical Analysis of Reliability and Life Data*, John Wiley & Sons, New York.

Mee, R. W. and Kushary, D. (1994) "Prediction Limits for the Weibull Distribution Utilizing Simulation", *Computational Statistics & Data Analysis*, 17, 327-336.

Patel, J. K. (1989) "Prediction Intervals - A Review", *Communications in Statistics - Theory and Methods*, 13, 2393-2465.

Pyke, R. (1965). "Spacings", *Journal of the Royal Statistical Society, Series B*, Vol 27, 395-449

APPENDIX

Invariance of Simulation Based Coverage Probabilities With Respect to the Shape Parameter

As discussed in Section 7, we estimated the theoretical coverage probabilities of the proposed prediction via a Monte Carlo simulation study. In this simulation, we generated Weibull random variables such that the stream of random variates for each parameter combination were initiated using the same seed. We will show that the coverage probabilities thus obtained are invariant with respect to β, provided that proportion of censored units in Type II schemes or the probability of a test unit lasting more than the censoring time T in Type I cases, is kept constant. This allows us to limit our Monte Carlo simulations to the $\beta = 1$ case because results would be identical for other values of β. We will first prove a result needed to establish this claim and then provide proofs for the Type I and Type II cases. Throughout this section we will assume, as was done in the simulation, that θ, the scale parameter of the underlying Weibull distribution, is equal to one.

Two features, one inherent to our simulation study and the other arising out of distributional theory, are the main factors that gives rise to the invariance property we will establish. First, in order to obtain a fair comparison of the performance of proposed bounds across various parameter combinations, and avoid confounding sample differences with the effect of parameter values on coverage properties, the same seed number was used when generating pseudo random variates for each such combination. Second, we have the property that if X~Weibull(1, ν), then X^{ν}~Exponential(1). Thus, the stream of random variates we would obtain from a Weibull (1,1) distribution is related to the stream of random variates we would obtain from a Weibull(1, ν) distribution, provided that the same seed was used in generating the two sets of random variates. The first stream can be obtained by raising the values in the second stream to the ν th power. We will formalize this concept in the following.

Suppose x_i, i= 1,2,…, r are the simulated values we obtain as the first r ordered observations from a past sample of size n from a Weibull (1, β) distribution with $\beta = 1$. For any positive constant ν, let $x_i^* = x_i^{1/\nu}$, i= 1,2,…, r. Then, we would obtain these very same x_i^* values as the first r ordered observations if we simulated data from a Weibull (1, β^*) distribution, where $\beta^* = \nu$, provided the same seed is used to initiate the random number generator.

Suppose we use x_i, i= 1,2,…, r to estimate the shape parameter β. Let $\hat{\beta}_S$ and $\hat{\beta}_{MLE}$ denote the simple estimator and the MLE of β. Similarly, let $\hat{\beta}_S^*$ and $\hat{\beta}_{MLE}^*$ be the simple estimator and the MLE of β^* based on x_i^*, i= 1,2,…, r. Then,

$$\hat{\beta}_S^* = \nu \hat{\beta}_S \qquad \text{(A.1)}$$

and

$$\hat{\beta}_{MLE}^* = \nu \hat{\beta}_{MLE}\,. \qquad \text{(A.2)}$$

The equality (A.1) follows because for $r<n$,

$$\hat{\beta}_S^* = \frac{nk_{r,n}^*}{\sum_{i=1}^{r-1}\left(y_r^* - y_i^*\right)} = \frac{nk_{r,n}^*}{\nu^{-1}\sum_{i=1}^{r-1}\left(y_r - y_i\right)} = \nu \hat{\beta}_S$$

where $y_i^* = ln(x_i^*) = \nu^{-1} ln(x_i) = \nu^{-1} y_i$ for i=1,2,….,r, and when $r=n$,

$$\hat{\beta}_S^* = \frac{nk_n^*}{-\sum_{i=1}^{s} y_i^* + \left(\frac{s}{n-s}\right)\sum_{i=s+1}^{n} y_i^*} = \frac{nk_n^*}{-\nu^{-1}\sum_{i=1}^{s} y_i + \left(\frac{s}{n-s}\right)\nu^{-1}\sum_{i=s+1}^{n} y_i} = \nu\hat{\beta}_S .$$

The result (A.2) follows from the following argument. First consider the Type II case.

Let $x_i, i = 1, 2, ..., r$ be ordered observations from a Weibull$(1, \beta)$ distribution. Then $\hat{\beta}$, the MLE of β, is given by the solution to the likelihood equation

$$(r+1)\beta^{-1} + \sum_{i=1}^{r} \ln x_i - \sum_{i=1}^{r} x_i^{\beta} \ln(x_i) - (n-r)(x_r)^{\beta} \ln(x_r) = 0.$$

Now let $x_i^* = x_i^{1/\nu}$ where $\nu > 0$, so x_i^*, $i = 1, 2, ..., r$ are the ordered observations from a Weibull$(1, \beta^*)$ distribution, where $\beta^* = \nu\beta$. Then $\hat{\beta}^*$, the MLE of $\beta^* = \nu\beta$, based on x_i^* i=1,2,…,n is given by the solution to

$$(r+1)(\beta^*)^{-1} + \sum_{i=1}^{r} ln(x_i^*) - \sum_{i=1}^{r} (x_i^*)^{\beta^*} ln\left(x_i^*\right) - (n-r)\left(x_r^*\right)^{\beta^*} ln\left(x_r^*\right) = 0$$

It is easily seen that $\hat{\beta}^* = \nu\hat{\beta}$ is a solution to the above equation by rewriting it as

$$(r+1)(\nu\beta)^{-1} + \sum_{i=1}^{r} \nu^{-1} ln(x_i) - \sum_{i=1}^{r} (x_i^{1/\nu})^{\nu\beta} \nu^{-1} ln(x_i) - (n-r)\left(x_r^{1/\nu}\right)^{\nu\beta} \nu^{-1} \ln(x_r) = 0.$$

Result for the Type I case follows by a similar argument.

We will from here on use the generic relation $\hat{\beta}^* = \nu\hat{\beta}$ in place of (A.1) and (A.2), with the implied understanding that both $\hat{\beta}^*$ and $\hat{\beta}$ are obtained using the same method of estimation.

Type I Case

Let the life-spans of n units with Weibull(1, β), $\beta = 1$, distribution be generated and Type I censored at time T. It is assumed that the censoring time T was selected so that $P(Z > T) = \gamma$ for $0 < \gamma < 1$, where Z ~Weibull(1, 1). Suppose that R, the number of failures by time T, was observed to be r, and x_i, i= 1,2,…, r are the ordered observations obtained under this censoring scheme, where $r \le n$.

For the Type I case, the (1-p)100% prediction limits are of the form

$$\hat{z}_p = \hat{\alpha}^{1/\hat{\beta}} \left\{ (1-p)^{-1/r} - 1 \right\}^{1/\hat{\beta}} \tag{A.1}$$

for lower bounds and

$$\hat{z}_p = \hat{\alpha}^{1/\hat{\beta}} \left\{ (p)^{-1/r} - 1 \right\}^{1/\hat{\beta}} \tag{A.2}$$

for upper bounds, where

$$\hat{\alpha} = \left\{ \left(\sum_{i=1}^{r} x_i^{\hat{\beta}} \right) + (n-r)T^{\hat{\beta}} \right\} \tag{A.3}$$

and $\hat{\beta}$ is the simple estimate (or MLE) of β. Observe that the coverage probability of the lower bound is given by,

$$P(Z \ge \hat{z}_p) = \exp\left\{ -\left(\frac{\hat{z}_p}{\theta} \right)^{\beta} \right\},$$

where Z ~ Weibull(1,1).

From the above, we obtain

$$P(Z \geq \hat{z}_p) = \exp\left\{-\hat{z}_p\right\}$$

$$= \exp\left[\left\{\left[\left(\sum_{i=1}^{r} x_i^{\hat{\beta}}\right) + (n-r)T^{\hat{\beta}}\right]^{\frac{1}{\hat{\beta}}}\right\}\left\{\left[(1-p)^{-\frac{1}{r}} - 1\right]^{\frac{1}{\hat{\beta}}}\right\}\right]. \quad \text{(A.4)}$$

Recall that we selected censoring time T such that $P(Z > T) = \gamma$. This implies that

$$P(Z > T) = \exp\left\{-T\right\} = \gamma$$

and therefore

$$T = -ln(\gamma).$$

Substituting this result in (A.4) we obtain

$$P(Z > \hat{z}_p) = \exp\left[\left\{\left[\left(\sum_{i=1}^{r} x_i^{\hat{\beta}}\right) + (n-r)(-ln(\gamma))^{\hat{\beta}}\right]^{\frac{1}{\hat{\beta}}}\right\}\left\{\left[(1-p)^{-\frac{1}{r}} - 1\right]^{\frac{1}{\hat{\beta}}}\right\}\right]. \quad \text{(A.5)}$$

Now consider simulating the case where n tested units have a Weibull(1, β^*) distribution where $\beta^* = \nu$. Then, by the results given earlier, the simple estimator (or MLE) of β^* is equal to $\nu\hat{\beta}$. Further, in the simulation study, the new censoring time T^* was selected so that $P(Z^* > T^*) = \gamma$, where Z^*~Weibull(1, ν). This ensures that if r out of n tested items had life-spans less than T when simulating from a Weibull(1,1) distribution, then exactly r out of n tested items will have life-spans less than T^* when data from a Weibull(1, ν) distribution are generated using the same seed. As before, the prediction limits, say $\hat{z}_p^*$, are given by (A.1) and (A.2), with $\hat{\beta}$ in (A.1), (A.2), and (A.3) replaced by $\nu\hat{\beta}$ and x_i in (A.3) replaced by $x_i^* = x_i^{1/\nu}$. Then the coverage probability of the lower bound is given by

$$P(Z^* > \hat{z}_p^*) = \exp\{-(\hat{z}_p^*)^{\nu}\}$$

$$= \exp\left[\left\{\left[\left(\sum_{i=1}^{r}(x^*)_i^{\hat{\beta}^*}\right) + (n-r)T^{*\hat{\beta}^*}\right]^{\frac{\nu}{\hat{\beta}^*}}\right\}\left\{\left[(1-p)^{-\frac{1}{r}} - 1\right]^{\frac{\nu}{\hat{\beta}^*}}\right\}\right].$$

$$= \exp\left[\left\{\left[\left(\sum_{i=1}^{r}(x_i^{1/\nu})^{\nu\hat{\beta}}\right) + (n-r)T^{*\nu\hat{\beta}}\right]^{\frac{\nu}{\nu\hat{\beta}}}\right\}\left\{\left[(1-p)^{-\frac{1}{r}} - 1\right]^{\frac{\nu}{\nu\hat{\beta}}}\right\}\right]. \qquad \text{(A.6)}$$

We selected censoring time T^* such that $P(Z^* > T^*) = \gamma$. This implies that

$$P(Z^* > T^*) = \exp\left\{-(T^*)^{\nu}\right\} = \gamma$$

and therefore

$$T^* = [-ln(\gamma)]^{1/\nu}.$$

Substituting this result in (A.6) we obtain,

$$P\left(Z^* > \hat{z}_p^*\right) = \exp\left[\left\{\left[\left(\sum_{i=1}^{r} x_i^{\hat{\beta}}\right) + (n-r)[-ln(\gamma)]^{\hat{\beta}}\right]^{\frac{1}{\hat{\beta}}}\right\}\left\{\left[(1-p)^{-\frac{1}{r}} - 1\right]^{\frac{1}{\hat{\beta}}}\right\}\right],$$

which is identical to the probability given in (A.5). Similarly, we can show that

$P(Z < \hat{z}_p) = P(Z^* < \hat{z}_p^*)$ for the upper prediction limits $\hat{z}_p$ and $\hat{z}_p^*$.

Type II Case

The proof for the Type II censored case follows in very similar fashion because the only difference in the prediction limits for the Type II case is that $T^{\hat{\beta}}$ in (A.3) is replaced by $z_{(r)}^{\hat{\beta}}$ and $z_{(r)}^{\hat{\beta}} = [z_{(r)}^*]^{\hat{\beta}^*}$.