# Quantifying the Information from a Randomized Response

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**Abstract**

Randomized response is a method designed for obtaining usable data on sensitive issues while protecting the privacy of those surveyed. Textbooks on introductory statistics, mathematical statistics, and sampling theory, among others, include examples of these techniques. But the impression that a randomized response completely hides sensitive information about the respondent is incorrect. In this paper, we look to analyze the information on a respondent divulged from such a survey.

**Key Words:** survey sampling, relative risk, Bayesian analysis

## 1. Introduction

Randomized response is a method designed for obtaining usable data on sensitive issues while protecting the privacy of those surveyed. Assuming each member in the population of interest belongs either to a sensitive group $A$ or its complement $A^c$, the problem is to estimate $\pi$, the proportion belonging to group $A$. If the question of group membership is asked directly, complete information on a respondent, in regards to their group membership, will clearly be divulged upon their answer. Thus, their privacy is not honored.

One technique is to have the respondent randomly select from the questions I:"Do you belong to group $A$" (chosen with probability $k$) and II:"Do you belong to group $A^c$" (chosen with probability $1-k$), responding only to the question selected. The interviewer observes the response but does not know which question is being answered. The probability of a $Yes$ response is given as

$$p = P(Yes) = k\pi + (1-k)(1-\pi). \tag{1.1}$$

From a sample of responses, an estimate of $p$ can be constructed. This will lead to an estimate of $\pi$.

Randomized response is an interesting application of elementary probability theory. Textbooks on introductory statistics (Freund,1988), mathematical statistics (DeGroot,1986), and sampling theory

(Tryfos,1996), among others, include examples of these techniques. Instructors can easily motivate ideas by actually taking a randomized response survey from the class. Students should not, however, get the impression that a randomized response completely hides all information about possible inclusion of the respondent in the sensitive group $A$. An argument which quantifies the disclosed information is easy to make and can be incorporated into a classroom discussion. This is presented in Section 2. An example is given in Section 3 where randomized response is analyzed from a Bayesian point of view using slightly more advanced techniques.

## 2. Quantifying Information

We define $k = P(I)$, $\pi = P(A)$, and $p = P(Yes)$. Let $x$ denote an observation from $BIN(n, p)$ and define $\hat{p} = x/n$. An unbiased estimate of $\pi$ is provided by

$$\hat{\pi} = \frac{\hat{p} + k - 1}{2k - 1} \tag{2.1}$$

with

$$Var(\hat{\pi}) = \frac{p(1-p)}{n(2k-1)^2}. \tag{2.2}$$

The choice of $k$ is fixed by the experimenter although we can assume without loss of generality that $\frac{1}{2} < k < 1$.

The two goals of a randomized response survey are to obtain as much information as possible on the value of $\pi$ while protecting as much information as possible on the specific classification of a respondent. A subject's response is either $Yes$ or $No$. So, the information divulged by a subject regarding membership in group $A$ is characterized through $P(A|Yes)$ and $P(A|No)$. These are calculated as

$$P(A|Yes) = \frac{k\pi}{p} = \frac{k\pi}{k\pi + (1-k)(1-\pi)} \tag{2.3}$$

and

$$P(A|No) = \frac{(1-k)\pi}{1-p} = \frac{(1-k)\pi}{(1-k)\pi + k(1-\pi)}. \tag{2.4}$$

Borrowing a term from the biostatistics literature, we define the **relative risk** of belonging to group $A$ from a $Yes$ response compared to a $No$ response as

$$R = \frac{P(A|Yes)}{P(A|No)}. \tag{2.5}$$

There is no information on a respondent when $R = 1$. As $R$ diverges away from 1, the information disclosed on a respondent's membership in the sensitive group $A$ increases. The goals of a randomized response survey design can now be quantified as a desire for small $Var(\hat{\pi})$ and small $|R-1|$.

The following results show how the ability to satisfy the above goals is affected by the choice of $k$.

**Proposition 2.1:** *$Var(\hat{\pi})$ is a decreasing function of $k$ on $\frac{1}{2} < k < 1$.*

**Proof :** Recall from (1.1), $p = k\pi + (1-k)(1-\pi)$. If $k = \frac{1}{2}$, then $p = \frac{1}{2}$. As $k$ increases from $\frac{1}{2}$, $p$ converges away from $\frac{1}{2}$ towards $\pi$. Since $p(1-p)$ decreases as $p$ moves away from $\frac{1}{2}$, the numerator in $Var(\hat{\pi})$ is decreasing in $k$. Clearly, the denominator $(2k-1)^2$ is increasing on $\frac{1}{2} < k < 1$. Therefore, $Var(\hat{\pi})$ is decreasing as a function of k, for $\frac{1}{2} < k < 1$. $\|$

**Proposition 2.2:** *$R$ is an increasing function of $k$ on $\frac{1}{2} < k < 1$.*

**Proof :** Define the functions $h(k) = k\pi$ and $g(k) = k\pi + (1-k)(1-\pi)$. The numerator in $R$ is $P(A|Yes) = h(k)/g(k)$. Note that $h$ and $g$ are linear with $h' = \pi$ and $g' = 2\pi - 1$. For $\frac{1}{2} < k_1 < k_2 < 1$, write

$$\frac{h(k_2)}{g(k_2)} = \frac{h(k_1) + (k_2 - k_1)h'}{g(k_1) + (k_2 - k_1)g'}.$$

Since $0 < h(k_1) < g(k_1)$ and $h' > g'$, we have $h(k_2)/g(k_2) > h(k_1)/g(k_1)$. Thus, the numerator in $R$ is increasing in $k$. Similarly, one can show that its denominator $P(A|No)$ is decreasing. Therefore, $R$ is increasing in $k$ for $\frac{1}{2} < k < 1$. $\|$

From Proposition 2.2 and the fact that $R(\frac{1}{2}) = 1$, we see $R(k) > 1$ for $\frac{1}{2} < k < 1$. A randomized response will always contain some information about the respondent, with the amount of information increasing as $k$ increases.

Tables 1 and 2 provide computed values of the quantities $R$ and $nVar(\hat{\pi})$ for various choices of $k$ and $\pi$. Choosing $k$ to be near 1, say .9, gives small variance for $\hat{\pi}$, but does not protect privacy. For example, if the true proportion in group $A$ is near .1, a $Yes$ response is around 41 times more likely to be in $A$ than a $No$ response. By looking at the problem in this light, one can see quantitatively how the privacy of a respondent has not been satisfied.

As our intuition suggests, the goals of minimizing $R$ and minimizing $Var(\hat{\pi})$ are in conflict. More-

Table 1: Values of $R$ as a function of $k$ and $\pi$

| k | $\pi = .1$ | $\pi = .3$ | $\pi = .5$ | $\pi = .7$ | $\pi = .9$ |
|---|---|---|---|---|---|
| .6 | 2.071 | 1.761 | 1.500 | 1.278 | 1.086 |
| .7 | 4.529 | 3.222 | 2.333 | 1.690 | 1.202 |
| .8 | 11.385 | 6.526 | 4.000 | 2.452 | 1.405 |
| .9 | 41.000 | 17.471 | 9.000 | 4.636 | 1.976 |

Table 2: Values of $nVar(\hat{\pi})$ as a function of $k$ and $\pi$

| k | $\pi = .1, .9$ | $\pi = .3, .7$ | $\pi = .5$ |
|---|---|---|---|
| .6 | 6.090 | 6.210 | 6.250 |
| .7 | 1.403 | 1.523 | 1.563 |
| .8 | 0.534 | 0.654 | 0.694 |
| .9 | 0.231 | 0.351 | 0.391 |

over, as $k \to 1$, $R \to \infty$ and as $k \to \frac{1}{2}$, $Var(\hat{\pi}) \to \infty$. Neither goal can be met completely without fully sacrificing the other. Any choice of $k$ other than $\frac{1}{2}$ or 1 represents a compromise.

## 3. Bayesian Approach

We look to investigate further the quantifying of information divulged by a respondent using techniques which are slightly more advanced, yet not beyond what is developed in a first course on mathematical statistics. Since we are dealing with the concept of measuring uncertainty of information, a Bayesian analysis seems appropriate.

Assume that the prior on $\pi$ is $Beta(a, b)$. Thus,

$$f(\pi) \propto \pi^{a-1}(1-\pi)^{b-1}, 0 < \pi < 1. \tag{3.1}$$

Standard transformation techniques can be used to show the induced prior on $p$ is given by

$$f(p) \propto [p - (1-k)]^{a-1}[k-p]^{b-1}, 1-k < p < k. \tag{3.2}$$

Our data consist of an observation $x$ from $BIN(n, p)$, where $p = k\pi + (1-k)(1-\pi)$, so the distribution on $\pi$ updates to

$$f(\pi|x) \propto [k\pi + (1-k)(1-\pi)]^x[(1-k)\pi + k(1-\pi)]^{n-x}\pi^{a-1}(1-\pi)^{b-1}, 0 < \pi < 1, \tag{3.3}$$

and the posterior on $p$ becomes

$$f(p|x) \propto p^x(1-p)^{n-x}[p-(1-k)]^{a-1}[k-p]^{b-1}, 1-k < p < k. \tag{3.4}$$

As an example, we consider a problem from Freund (1988). The sensitive group of interest is defined by the statement "I smoke marijuana at least once a week". A sample of 250 students was selected for a randomized response survey with $k = .6$. Of those students, $x = 106$ answered yes. Then $\hat{p} = 106/250 = .424$ and, from (2.1), $\hat{\pi} = (\hat{p} - .4)/.2 = .12$.

Assuming a uniform prior for $\pi$ is reasonable and will simplify the mathematics. The posterior on $p$ can be determined from (3.4) by taking $a = b = 1$. See that $p$ here follows a $Beta(107, 145)$ restricted to the interval $(.4, .6)$. Then the cumulative distribution function of $p$ is

$$F(p|x) = \frac{\beta(p) - \beta(.4)}{\beta(.6) - \beta(.4)}, \tag{3.5}$$

where $\beta$ is used to denote the cdf of $Beta(107, 145)$. Percentiles of Beta distributions are readily available from a statistical software package such as Minitab.

An 80% Bayesian confidence interval for $p$ can be found by solving $F(p_L|x) = .1$ and $F(p_U|X) = .9$ as $(p_L, p_U) = (.4076, .4688)$. Then

$$(\pi_L, \pi_U) = (\frac{p_L - .4}{.2}, \frac{p_U - .4}{.2}) = (.038, .344) \tag{3.6}$$

defines an interval estimate for $\pi$.

The information divulged by a respondent can also be analyzed within this Bayesian framework. Since $R = (\frac{k}{1-k})(\frac{1-p}{p})$, an interval estimate computes to be

$$(R_L, R_U) = (\frac{.6(1-p_U)}{.4(p_U)}, \frac{.6(1-p_L)}{.4(p_L)}) = (1.70, 2.18). \tag{3.7}$$

So we can believe with good strength (probability .8) that a subject responding $Yes$ is between 1.70 and 2.18 times more likely to "smoke marijuana at least once a week" than a subject responding $No$.

Because $k$ is near $\frac{1}{2}$, the conditions for a randomized response design protecting some degree of privacy are met. However, the wide interval for $\pi$ shows how the accuracy of estimation suffers.

## 4. Concluding Remarks

We consider in this paper a simple randomized response technique. For a discussion of more extensive models, the reader is referred to Chaudhuri and Mukerjee (1988).

The purpose here is not to attain results which are beyond intuition, but to introduce an easily understood setting where one can study the effectiveness of randomized response as a tool for masking the characteristics of a respondent.

## References

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