# Generalizing Galileo's Passedix Game

BY

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**Abstract**

Passedix (translates as passten) is a game of chance; it is played by three fair dice. In the present paper a description and analysis of the game as well as a generalization will be undertaken.

**Keywords:** Teaching; Passedix game; Generalization

## 1. Introduction

Rolling the dice is a highly favoured activity in classrooms introducing students into the laws of probability. Rolling the dice using three or more of them creates complications; counting the outcomes in the algebra of events that emerge from the given sample space becomes a tedious task. While it is straightforward to analyze simple examples that come from a sample space of tossing two dice, students invariably find difficulties in generalizing counting techniques. Complexity in combinations on the one hand and permutations on the other has been problematic in teaching probabilities and remains so. Leibniz (1646-1716) – whom Newton (see Schell 1960) had called father of calculus – was mistaken for not having taken the ordering into account. Leibniz's mistake or omission if you will, still dominates modern mathematical thinking. Since Glickman (1990, 1991), we are aware of Leibniz's statement, according to which there is a unique way to produce a partition of total 12, if we toss two fair dice, that is 6+6 and only one way to partition total 11 (6+5).

Leibniz (see Todhunter 1865) was of the opinion that "…with two dice, it is as feasible to throw a total 12 points as to throw a total 11: as either one can only be achieved in one way…". It is well known today that in the experiment of tossing two fair dice, P [sum 11] = P [sum 3] = 2/36, whereas P [sum 12] = P [sum 2] = 1/36 etc, since the subsums with the sum 14 have all equal probabilities.

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According to Monmort (1678-1719), most of combinatorics came from Pascal (1623-1662) and Pascal and Fermat (1601-1665) are considered to be the “joint discoverers” of the probability calculus. Pascal gives us a clear review of James Bernoulli's work, with many references to Leibniz as well. In the 16$^{th}$ century Girolamo Gardano (1501-1576), who was involved in gambling, had stated quite clearly in his work on gambling games (Liber De Ludo Aleae) correcting Leibniz’ s argument. He says: “in the case of two dice, the points of total 12 and 11 can be obtained respectively as (6,6) and (6,5)…. but the latter can occur in two ways”.

The purpose of the present paper is to generalize the game of chance which is known as “passedix”. The word is French and it means “passten”; we are of the opinion that the game was born following one of Galileo’s (1898) explanations on a paradox that occurred in the experiment of tossing three dice.

## 2. A Historical Background and the Description of the Game

In the first part of the 17$^{th}$ century Galileo Galilei (1564-1642) was born, the same year as Shakespeare, in the Italian town of Pisa. He was sent to the University of Pisa to study medicine, but it was a professorship of mathematics that he was elected in 1589 (at the age of 25) through the favours of Ferdinando dei Medici, the Grand Duke of Tuscany. His manuscripts and papers were all burnt by his daughter after his death. Galileo, the most prominent natural scientist and astronomer of his time was “ordered” by the Grand Duke of Tuscany to explain a paradox arising in the experiment of tossing three dice. The following statement occurs in *Thoughts about Dice Games*: “now I, in order to oblige him (i.e. the Grand Duke) who has ordered me to produce whatever occurs to me about the problem, will expound my ideas …”. David (1962) says: “It is a little difficult to imagine anyone ordering Galileo to do anything (and being obeyed) except the person who paid him”.

Galileo’ s style of writing is noteworthy for its clarity and also for its love of brevity and conciseness. Galileo always plunged straight into the argument. The Duke asked him to explain: “Why, although there were an equal number of 6 partitions of the numbers 9 and 10, did experience state that the chance of throwing a total 9 with three fair dice was less than that of throwing a total of 10?” The above problem and the paradoxes arising therefrom consisted of seeing only partitions of the various totals that the dice could generate and ignoring ordering, or to put it in another way, not distinguishing between the dice when calculating these totals.

Laplace (1812) after his a priori definition of probability said: “the exact estimation of various cases is one of the important points of Analysis of

Chance". The estimation of various partitions in games of chance has been accurately achieved by Galileo, describing these, in the three fair dice experiment, as sampling with order. He noticed that each of the numbers 9, 10, 11 and 12 are possible to come up as a sum of three addenda. Writing these in an increasing order, he obtained the following table (1):

| | | | |
|---|---|---|---|
| 1+4+6=11 | 1+5+6=12 | 3+3+3=9 | 1+3+6=10 |
| 2+3+6=11 | 2+4+6=12 | 1+2+6=9 | 1+4+5=10 |
| 2+4+5=11 | 3+4+5=12 | 1+3+5=9 | 2+2+6=10 |
| 1+5+5=11 | 2+5+5=12 | 1+4+4=9 | 2+3+5=10 |
| 3+3+5=11 | 3+3+6=12 | 2+2+5=9 | 2+4+4=10 |
| 3+4+4=11 | 4+4+4=12 | 2+3+4=9 | 3+3+3=10 |

**Table 1:** Partitions of sums 11, 12, 9 and 10 of the game of three fair dice.

From Table (1) he realized that each of the above sums came up in six different ways. It was obvious, to him, that the partitions in question were not equivalent. Indeed, if we consider, for example the sum 9, this can come up in only one way, e.g., from 3, 3, 3; whereas from the addenda 1, 2, 6 in six different ways; from 2, 2, 5 in three ways etc. Having all the above at hand, Galileo drew up the following Table (2):

| **Sum: 9** | **cases** | **Sum: 10** | **cases** | **Sum: 11** | **cases** | **Sum: 12** | **cases** |
|---|---|---|---|---|---|---|---|
| 126 | 6 | 136 | 6 | 146 | 6 | 156 | 6 |
| 135 | 6 | 145 | 6 | 155 | 3 | 246 | 6 |
| 144 | 3 | 226 | 3 | 236 | 6 | 255 | 3 |
| 225 | 3 | 235 | 6 | 245 | 6 | 336 | 3 |
| 234 | 6 | 244 | 3 | 335 | 3 | 345 | 6 |
| 333 | 1 | 334 | 3 | 344 | 3 | 444 | 1 |
| **Total** | **25** | | **27** | | **27** | | **25** |

**Table 2:** Permutations of the game of three fair dice.

Table (2) reveals that a sum 9 is less frequent than a sum 10 and a sum 11 is more frequent than a sum 12 and this was also the opinion of his friend and patron, the Duke of Tuscany. Also, he noticed that summations with sum 21 had the same probability. Moreover, he observed that

$$P\,[\text{sum} > 10] = P\,[\text{sum} \leq 10] = 108/216 = 50\%$$

The above equation was the origin of a game of chance named "passedix" as the ancient Greek astragaloi had been the origin of the games of dice. Player A, for example, wins if he rolls a sum above 10 and loses otherwise. Betting some amount of money they throw sequentially three fair dice. The winner takes the money.

## 3. Generalization

Lemma

Instead of tossing three fair dice, let us throw an odd number of ($n$) dice. I will prove that:

(a) $\text{Prob}\left[\text{Sum} > (7n-1)/2\right] = 1/2$

(b) $\text{Prob}\left[\text{Two subsums add up to } 7n\right]$ are all equal. (i.e. the probabilities of any two subsums to add up to $7n$ are all equal).

PROOF (a) Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the indications of $n$ (odd) tossed fair dice. Then, the following inequality holds

$$\alpha_1 + \alpha_2 + \ldots + \alpha_n > (7n-1)/2 \quad (n = \text{odd}) \Leftrightarrow \tag{1}$$

$$\Leftrightarrow 7 - \alpha_1 + 7 - \alpha_2 + \ldots + 7 - \alpha_n < (7n+1)/2 . \tag{2}$$

Hence, $$7 - \alpha_1 + 7 - \alpha_2 + \ldots + 7 - \alpha_n \leq (7n+1)/2 - 1 = (7n-1)/2 , \tag{3}$$

since $7 - \alpha_i$, $i = 1, 2, \ldots, n$ are the underneath indications of our dice. Subsequently, we observe that to each favourable partition (1) corresponds only one unfavourable one, that of (3) and inversely. Therefore, Laplace would say that the probability having a sum greater than $(7n-1)/2$ is $1/2 = 50\%$ .

PROOF (b) Let suppose that $\sum \alpha_i = S$ , $\sum \alpha'_i = S'$ and $S + S' = 7n$

where $\alpha_i, \alpha'_i$, $i = 1, 2, \ldots, n$ are the up and down indications of our dice respectively. Then,

$$\text{Prob}[S'] = \text{Prob}\left[\sum \alpha'_i\right] = \text{Prob}\left[\sum (7 - \alpha'_i)\right] \text{ (by part a)}$$
$$= \text{Prob}[7n - S']$$
$$= \text{Prob}[S] \text{ (by assumption)}$$

It turns out that (b) is true for every $n \in$ ¥ . This completes the proof.

Remark: Let $(7n-1)/2$ be called the "passpoint" of the players when they use $n$ (odd) dice. For example if $n=3$ then the passpoint is $(7\cdot 3-1)/2=10$, if $n=5$ then the players' passpoint will be $(7\cdot 5-1)/2=17$ etc. The following table (3) gives passpoints up to $n$ (odd) dice.

| No of dice | 1 | 3 | 5 | 7 | 9 | … | $n$ |
|---|---|---|---|---|---|---|---|
| Passpoints | 3 | 10 | 17 | 24 | 31 | … | $(7n-1)/2$ |

**Table 3:** Passpoints up to *n* (odd) dice.

We observe that the above sequence of passpoints consists of an arithmetic projection with general term $u_n=(7n-1)/2$ ($n$ is odd) and common difference 7. We know that in ancient times, number seven was considered to be a sacred number. We are of the opinion that this number has something special: It is no coincidence that we have seven days in a week, seven miracles of the world, seven hills in Rome, seven wise men of antiquity, seven eclipses per year, seven nobles elected in ancient Sparta. Herodotus' Histories provide extensive reference on this number. Mahabharata narrates that seven days lasted the cataclysm that Intra (God of Rain) brought upon the Indian people to punish them. It is noteworthy that if we use five dice the better name of the game should be, by tradition, "passedix-sept", which means passseventeen etc.

## 4. Conclusion

Although empirical experimentation arguably plays an important role in the development of probability theory, observation is of paramount importance in the decision making. Nowadays, with the help of computer simulation techniques, one can identify many of paradoxes in the field of mathematical games, using coins, urns and dice. For more information on paradoxes of games, see Hombas (1997) and the relevant references.

Sometimes we tend to believe that ideas do not come when we want them but at times when they are least needed. If this is the case the solution of problems would depend primarily on chance. Many people believe that this is the case. The poet Samuel Butler (1835) expressed this opinion in four witty lines:

> *"All the inventions that the world contains,*
> *Were not by reason first found out, nor brains;*
> *But pass for theirs who had the luck to light*
> *Upon them by mistake or oversight".*

We trust that the present paper will stimulate further debate on the quantitative aspects of the ancient world and mathematical games.

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