A GENERALIZED BETA DISTRIBUTION II

by

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ABSTRACT: A new distribution which contains the beta distribution as a special case is introduced. Several properties of the distribution (including its hazard rate function and moments) are derived.

1 INTRODUCTION

Beta distributions are very versatile and a variety of uncertainties can be usefully modeled by them. Many of the finite range distributions encountered in practice can be easily transformed into the standard distribution. In reliability and life testing experiments, many times the data are modeled by finite range distributions, see for example Barlow and Proschan (1975).

A random variable X is said to have the standard beta distribution with parameters a and b if its probability density function (pdf) is:

$$f(x) = \frac{x^{a-1}(1-x)^{b-1}}{B(a,b)} \tag{1}$$

for $0 < x < 1$, $a > 0$ and $b > 0$, where

$$B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1} dt$$

denotes the beta function. Many generalizations of (1) involving algebraic and exponential functions have been proposed in the literature; see Chapter 25 in Johnson *et al.* (1995) and Gupta and Nadarajah (2004) for detailed accounts. In this note, we introduce a generalization of (1) involving the Gauss hypergeometric function. The properties of this distribution (including its cdf, moments, hazard rate function and particular cases) are derived in Sections 2 and 3. The calculations of this note use the incomplete beta function and the hypergeometric functions defined by

$$B_x(a,b) = \int_0^x t^{a-1}(1-t)^{b-1} dt,$$

$$_2F_1\left(a,b;c;x\right) = \sum_{k=0}^{\infty} \frac{(a)_k\,(b)_k}{(c)_k} \frac{x^k}{k!},$$

and

$$ {}_3F_2\left(a, b, c; d, e; x\right) = \sum_{k=0}^{\infty} \frac{(a)_k\,(b)_k\,(c)_k}{(d)_k\,(e)_k} \frac{x^k}{k!}, $$

where $(z)_k = z(z+1)\cdots(z+k-1)$ denotes the ascending factorial. The first of the hypergeometric functions is known as the Gauss hypergeometric function. The properties of these special functions can be found in Prudnikov *et al.* (1990) and Gradshteyn and Ryzhik (2000).

2 THE DISTRIBUTION

We define the new distribution to have the pdf

$$ f(x) = \frac{bB(a,b)}{B(a,b+\gamma)} x^{a+b-1}\, {}_2F_1\left(1-\gamma, a; a+b; x\right) \tag{2} $$

for $0 < x < 1$, $a > 0$, $b > 0$ and $\gamma > 0$. The corresponding cdf is:

$$ F(x) = \frac{bB(a,b)}{B(a,b+\gamma)} \int_0^x y^{a+b-1}\, {}_2F_1\left(1-\gamma, a; a+b; y\right) dy. \tag{3} $$

By an application of equation (2.21.1.4) in Prudnikov *et al.* (1990, volume 3), (3) can be reduced to

$$ F(x) = \frac{bB(a,b)}{(a+b)B(a,b+\gamma)} x^{a+b}\, {}_2F_1\left(1-\gamma, a; a+b+1; x\right). \tag{4} $$

The nth moment associated with (2) is:

$$ E\left(X^n\right) = \frac{bB(a,b)}{B(a,b+\gamma)} \int_0^1 x^{n+a+b-1}\, {}_2F_1\left(1-\gamma, a; a+b; x\right) dx. \tag{5} $$

Now, by an application of equation (2.21.1.5) in Prudnikov *et al.* (1990, volume 3), (5) can be reduced to

$$ E\left(X^n\right) = \frac{bB(a,b)}{(n+a+b)B(a,b+\gamma)}\, {}_3F_2\left(1-\gamma, a, n+a+b; a+b, n+a+b+1; 1\right). $$

Note that in the particular case $b = 0$, (2) reduces to the standard beta pdf (1) with parameters a and γ. Figure 1 illustrates the shape of (2) for selected values of (a, b, γ). The effect of the parameter b can be clearly seen.

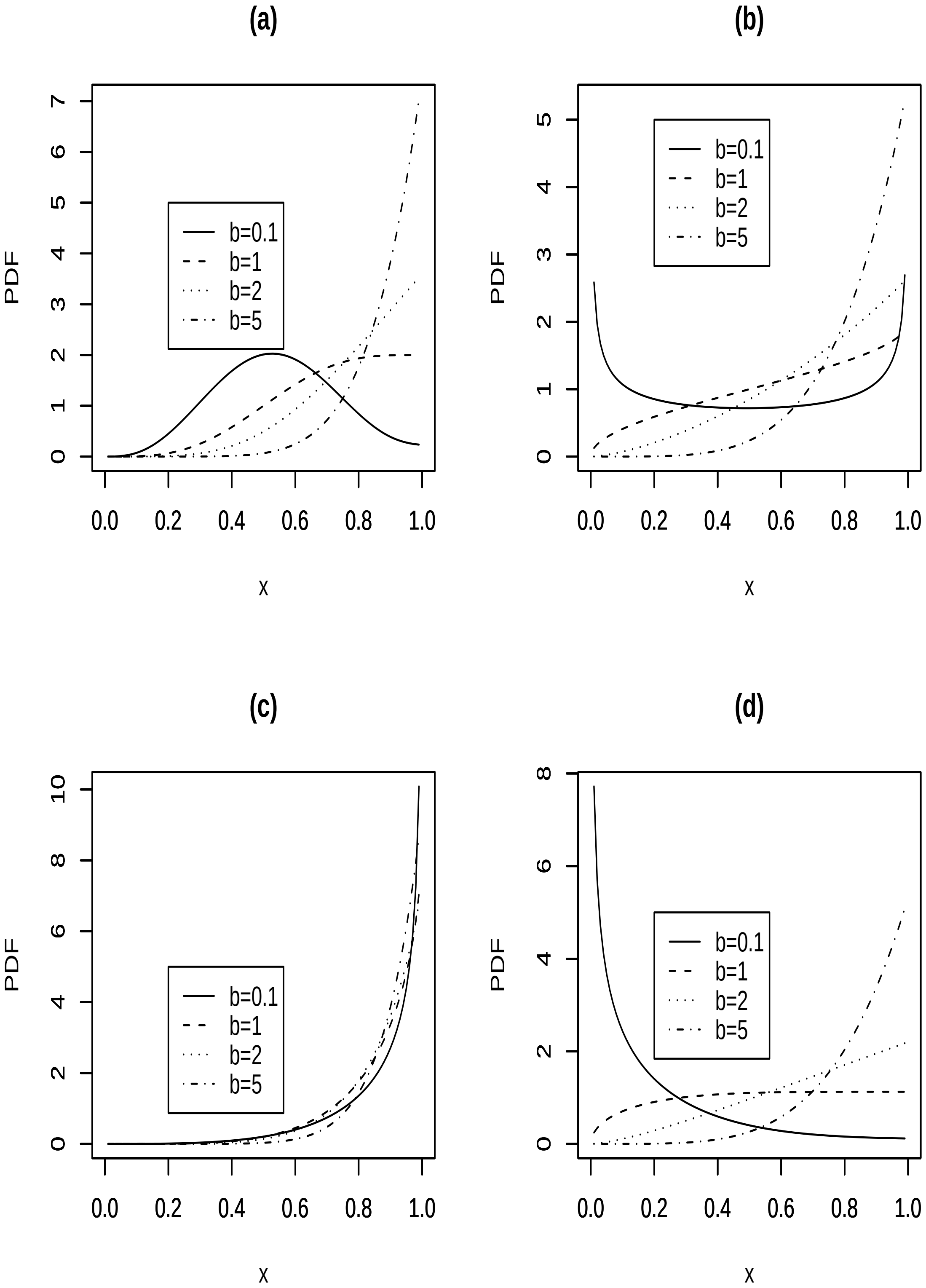

Figure 1. The pdf (2) for (a): $(a, \gamma) = (4, 4)$; (b): $(a, \gamma) = (0.5, 0.5)$; (c): $(a, \gamma) = (4, 0.5)$; and, (d): $(a, \gamma) = (0.5, 4)$.

The hazard rate function defined by $h(x) = f(x)/\{1 - F(x)\}$ is an important quantity characterizing life phenomena. It is immediate from (2) and (4) that the hazard rate function is given by

$$h(x) \quad = \quad \frac{b(a+b)B(a,b)x^{a+b-1}\ {}_2F_1\left(1-\gamma, a; a+b; x\right)}{(a+b)B(a,b+\gamma) - bB(a,b)x^{a+b}\ {}_2F_1\left(1-\gamma, a; a+b+1; x\right)}. \tag{6}$$

Some possible shapes of (6) for $a = \gamma = 1/2$ and $b = 0.1, 1, 2, 5$ are shown in Figure 2. Realistically, one would expect a 'bath-tubed' shape for $h(x)$, and it is pleasing to see that the shapes exhibited in Figure 2 are exactly of this type.

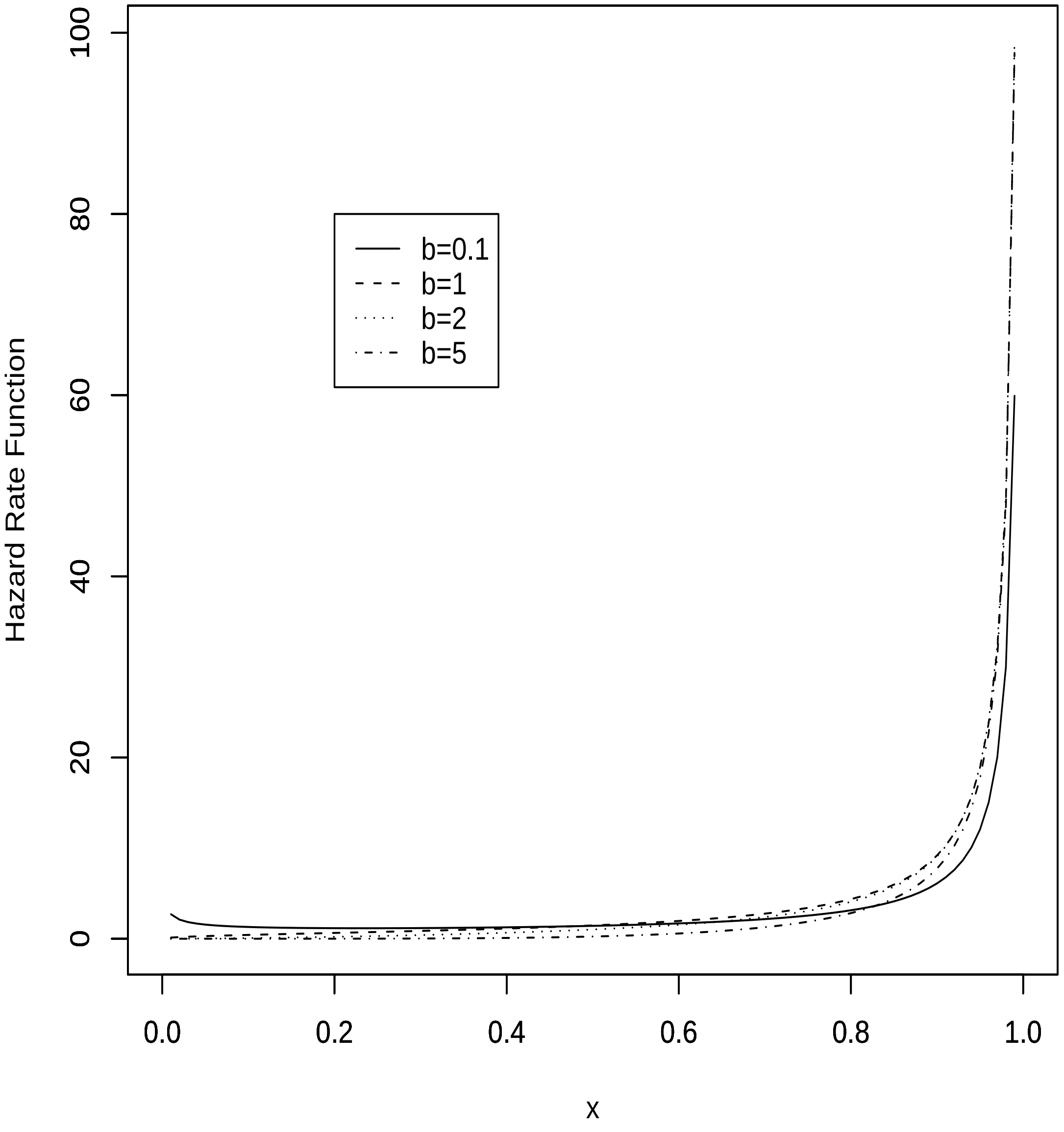

Figure 2. The hazard rate function (6) for $a = \gamma = 0.5$.

3 PARTICULAR CASES

Several particular cases of (2) can be obtained by using special properties of the Gauss hypergeometric function (see Section 7.3 of Prudnikov *et al.* (1990) and Sections 9.10 to 9.13 of Gradshteyn and Ryzhik (2000)). Some of these cases are:

1. If $a+b+\gamma=1$ then (2) reduces to

$$f(x) = \frac{b\Gamma(b)x^{a+b-1}(1-x)^{-a}}{\Gamma(1-a)\Gamma(a+b)}.$$

2. If $a+b+\gamma=2$ then

$$f(x) = \frac{b(a+b-1)B(a,b)}{B(a,2-a)}B_x(a+b-1,1-a).$$

If in addition

(i) $a+b-1$ is an integer then

$$\begin{aligned} f(x) &= \frac{b(a+b-1)B(a,b)B(a+b-1,1-a)}{B(a,2-a)} \\ &\quad \times \left\{1-\sum_{l=1}^{a+b-1}\frac{\Gamma(l-a)}{\Gamma(1-a)\Gamma(l)}x^{l-1}(1-x)^{1-a}\right\}. \end{aligned}$$

(ii) $a=1/2$ and $b=1$ then

$$f(x) = \frac{4}{\pi}\arctan\sqrt{\frac{x}{1-x}}.$$

(iii) $a=1/2$ and $b=k$ then

$$\begin{aligned} f(x) &= \frac{k(2k-1)B\left(1/2,k\right)B\left(1/2,k-1/2\right)}{\pi} \\ &\quad \times \left\{\frac{2}{\pi}\arctan\sqrt{\frac{x}{1-x}}-\sqrt{x(1-x)}\sum_{l=1}^{k-1}d_l\right\}. \end{aligned}$$

3. If $\gamma=0$ then

$$f(x) = b(a+b-1)(1-x)^{b-1}B_x(a+b-1,1-b).$$

If in addition

(i) $a+b-1$ is an integer then

$$\begin{aligned} f(x) &= b(a+b-1)B(a+b-1,1-b)(1-x)^{b-1} \\ &\quad \times \left\{1-\sum_{l=1}^{a+b-1}\frac{\Gamma(l-b)}{\Gamma(1-b)\Gamma(l)}x^{l-1}(1-x)^{1-b}\right\}. \end{aligned}$$

(ii) $a = 1$ and $b = 1/2$ then

$$f(x) = \frac{1}{2\sqrt{1-x}} \arctan \sqrt{\frac{x}{1-x}}.$$

(iii) $a = k$ and $b = 1/2$ then

$$f(x) = \frac{(2k-1)B\left(1/2, k-1/2\right)}{4\sqrt{1-x}} \left\{ \frac{2}{\pi} \arctan \sqrt{\frac{x}{1-x}} - \sqrt{x(1-x)} \sum_{l=1}^{k-1} d_l \right\}.$$

4. If $\gamma = 1$ then

$$f(x) = (a+b)x^{a+b-1},$$

a power function pdf.

5. If $a = 0$ then

$$f(x) = bx^{b-1},$$

another power function pdf.

6. If $b = 1$ then

$$f(x) = \frac{B_x(a, \gamma)}{B(a, \gamma + 1)}.$$

If in addition

(i) a is an integer then

$$f(x) = \left(\frac{a}{\gamma} + 1\right) \left\{ 1 - \sum_{l=1}^{a} \frac{\Gamma(\gamma + l - 1)}{\Gamma(\gamma)\Gamma(l)} x^{l-1} (1-x)^{\gamma} \right\}.$$

(ii) γ is an integer then

$$f(x) = \left(\frac{a}{\gamma} + 1\right) \left\{ \sum_{l=1}^{\gamma} \frac{\Gamma(a + l - 1)}{\Gamma(a)\Gamma(l)} x^{a} (1-x)^{l-1} \right\}.$$

(iii) $a = \gamma = 1/2$ then

$$f(x) = \frac{4}{\pi} \arctan \sqrt{\frac{x}{1-x}}.$$

(iv) $a = k - 1/2$ and $\gamma = j - 1/2$ then

$$f(x) = \left(\frac{a}{\gamma} + 1\right) \left\{ \frac{2}{\pi} \arctan \sqrt{\frac{x}{1-x}} - \sqrt{x(1-x)} \sum_{l=1}^{k-1} d_l + \sum_{l=1}^{j-1} c_l \right\}.$$

It should be noted above that the constants c_l and d_l are given by

$$c_l = \frac{\Gamma(k+l-1)x^{k-1/2}(1-x)^{l-1/2}}{\Gamma\left(k-1/2\right)\Gamma\left(l+1/2\right)}$$

and

$$d_l = \frac{\Gamma(l)x^{l-1}}{\Gamma\left(l+1/2\right)\Gamma\left(1/2\right)},$$

respectively.

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