Maximum Likelihood Estimation for Short Time Series with Replicated Observations: A Simulation Study

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Abstract

Analysis of a large number of independent replications from short, first order autoregressive type time series is considered. Maximum likelihood estimation (mle) procedure is discussed in both approximate and exact forms. A simulation study is carried out. It is shown that both the approximate and exact mle methods provide unbiased and very efficient (in the minimum mean square sense) estimates for the parameters.

Keywords: Time series; Correlation; Estimation; Bias; Repeated measurements; Efficient; Maximum likelihood; Minimum mean square

1 Introduction

It is known that there are many situations in practice, especially in medical research where one observes several very short time series (see, for instance, Cox and Solomon (1988) and Rai et. al. (1995)). Though the theory for time series is well developed to deal with series containing many observations, in this case one can not rely on the usual estimation or asymptotic theory. In a situation where one suspects the serial correlation among observations in a short realization, it is reasonable to begin with a standard autoregressive moving average (ARMA) type model given by

$$\Phi(B)X_t = \Theta(B)e_t, t = 1, \cdots, n \ , \tag{1.1}$$

where $\Phi(B) = I - \phi_1 B - \cdots - \phi_p B^p$ and $\Theta(B) = I - \theta_1 B - \cdots - \theta_q B^q$ are stationary autoregressive (AR) and invertible moving average (MA) polynomials in B (the backshift operator satisfying $B^j U_t = U_{t-j}$); I is the identity

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operator; $\{e_t\}$ is a sequence of uncorrelated random variables (not necessarily independent) with zero mean and variance σ^2. When n is small one can handle the situation by taking m (m is large) independent repeated measurements on (1.1). In this case write (1.1) as autoregressive moving average (ARMA) type model for $t = 1, \cdots, n$ given by

$$\Phi(B)X_{it} = \Theta(B)e_{it}, \tag{1.2}$$

where $i = 1, \cdots, m$. For simplicity, this paper considers the repeated measurements from a first order autoregressive model with a non-zero mean satisfying

$$(I - \phi B)(X_{it} - \mu) = e_{it}, i = 1, \cdots, m\ , \tag{1.3}$$

where $|\phi| < 1$ and μ are constants. Assume that ϕ and μ remain unchanged for each series. It is known that $\mu = E(X_{it})$.

This paper extends the maximum likelihood estimation (mle) procedures (exact and approximate) to estimate ϕ, μ and σ^2 as these parameters play an important role in many practical situations. With that view in mind, in Section 2 we review and extend the mle procedures in both exact and approximate cases. Section 3 reports the results from a simulation study.

2 Maximum Likelihood Analysis

2.1 Conditional Likelihood Estimation

Let$n$be the number of observations on X_{it} in each series for $i = 1, ., \cdots, m$ and let the series X_{it} is independent of the series X_{jt} for each $i \neq j$. Assuming e_{it} are independent and identically distributed (i.i.d.) $N(0, \sigma^2)$ (ie. $\{e_{it}\}$ is a Gaussian white noise), the corresponding conditional likelihood function based on (1.3) can be written as

$$L = \prod_{i=1}^{m} (\frac{1}{\sqrt{(2\pi)}\sigma})^{n-1} exp\{\frac{-1}{2\sigma^2} \sum_{t=2}^{n} (X_{it} - \mu - \phi(X_{i,t-1} - \mu))^2. \tag{2.1}$$

Maximizing the log likelihood, $ln(L)$ with respect to the parameters ϕ, μ, and σ^2, we have

$$\hat{\phi} = \frac{\sum_{i=1}^m \sum_{t=2}^n X_{it} \sum_{i=1}^m \sum_{t=2}^n X_{i,t-1} - m(n-1) \sum_{i=1}^m \sum_{t=2}^n X_{it}X_{i,t-1}}{(\sum_{i=1}^m \sum_{t=2}^n X_{i,t-1})^2 - m(n-1) \sum_{i=1}^m \sum_{t=2}^n X_{i,t-1}^2}, \quad (2.2)$$

$$\hat{\mu} = \frac{\sum_{i=1}^m \sum_{t=2}^n X_{it} \sum_{i=1}^m \sum_{t=2}^n X_{i,t-1}^2 - \sum_{i=1}^m \sum_{t=2}^n X_{i,t-1} \sum_{i=1}^m \sum_{t=2}^n X_{it}X_{i,t-1}}{(m(n-1) \sum_{i=1}^m \sum_{t=2}^n X_{i,t-1}^2 - (\sum_{i=1}^m \sum_{t=2}^n X_{i,t-1})^2)(1-\hat{\phi})}, \quad (2.3)$$

and

$$\hat{\sigma}^2 = \frac{\sum_{i=1}^m \sum_{t=2}^n (X_{it} - \mu - \hat{\phi}(X_{i,t-1} - \mu))^2}{m(n-1)}. \quad (2.4)$$

Note:

$$\hat{\mu} = \frac{\sum_{i=m}^m \sum_{t=2}^n (X_{it} - \hat{\phi}X_{i,t-1})}{m(n-1)(1-\hat{\phi})}.$$

Equations (2.2), (2.3) and (2.4) can be simplified using vector notation as follows:

Define V_1 as a row vector of 1's of order $1 \times m$ and V_2 as a column vector of 1's of order $(n-1) \times 1$. Let A be the $m \times n$ matrix of all (X_{it})'s for $i = 1, \cdots m; t = 1, \cdots n$ and let A_1 and A_2 be two matrices of order $m \times (n-1)$ excluding the first column and the last column of A respectively. Let $sum(A)$ be the sum of all elements of the matrix A. Then obviously, one has

$$\sum_{i=1}^m \sum_{t=2}^n X_{it} = V_1 A_1 V_2 = sum(A_1),$$

and

$$\sum_{i=1}^m \sum_{t=2}^n X_{i,t-1} = V_1 A_2 V_2 = sum(A_2).$$

Also,

$$\sum_{i=1}^m \sum_{t=2}^n X_{it} X_{i,t-1} = V_1 (A_1 * A_2) V_2 = sum(A_1 * A_2),$$

and

$$\sum_{i=1}^m \sum_{t=2}^n X_{i,t-1}^2 = V_1 (A_2 * A_2) V_2 = sum(A_2^2),$$

where $A * B$ denotes the matrix formed by the product of the 'corresponding elements of the matrices A and B and $A * A = A^2$. Now the corresponding conditional ml estimates can be written as

$$\hat{\phi} = \frac{(V_1A_1V_2)(V_1A_2V_2) - m(n-1)V_1(A_1 * A_2)V_2}{(V_1A_2V_2)^2 - m(n-1)V_1(A_2 * A_2)V_2} \tag{2.5}$$

$$= \frac{sum(A_1)sum(A_2) - m(n-1)sum(A_1 * A_2)}{sum(A_2)^2 - m(n-1)sum(A_2^2)} \tag{2.6}$$

$$\hat{\mu} = \frac{(V_1A_1V_2)(V_1(A_2 * A_2)V_2) - (V_1A_2V_2)(V_1(A_1 * A_2)V_2)}{(m(n-1)V_1(A_2 * A_2)V_2 - (V_1A_2V_2)^2)(1-\hat{\phi})} \tag{2.7}$$

$$= \frac{sum(A_1)sum(A_2^2) - sum(A_2)sum(A_1 * A_2)}{(m(n-1)sum(A_2^2) - sum(A_2)^2)(1-\hat{\phi})} \tag{2.8}$$

$$\hat{\sigma}^2 = (1-\hat{\phi}^2)\hat{\mu}^2 + P_1 + P_2, \tag{2.9}$$

where

$$P_1 = \frac{V_1(A_1 * A_1)V_2) + \hat{\phi}^2 V_1(A_2 * A_2)V_2}{m(n-1)} = \frac{sum(A_1^2) - \hat{\phi}^2 sum(A_2^2)}{m(n-1)}$$

and

$$P_2 = \frac{2\hat{\mu}(1-\hat{\phi})(\hat{\phi}(V_1A_2V_2) - (V_1A_1V_2)) - 2\hat{\phi}(V_1(A_1 * A_2)V_2)}{m(n-1)}.$$

$$= \frac{2\hat{\mu}(1-\hat{\phi})(\hat{\phi}sum(A_2) - sum(A_1)) - 2\hat{\phi}(sum(A_1 * A_2))}{m(n-1)}.$$

Using these equations (2.6), (2.8) and (2.9), it is easy to estimate the parameters based on simulated values from (1.3).

Denote the corresponding vector of the estimates by $\hat{\delta}_1$, where $\hat{\delta}_1 = (\hat{\phi}_1, \hat{\mu}_1, \hat{\sigma}_1^2)'$.

Now we look at the exact maximum likelihood estimation procedure in Section 2.2.

2.2 Exact Likelihood Estimation

Consider a stationary normally distributed AR(1) time series $\{X_{it}\}$ generated by (1.3). Let X_T be a sample of size $T = mn$ from (1.3) and let $X_T = (X_1, \cdots, X_m)'$, where $X_i = (X_{i1}, \cdots, X_{in})'$. In other words the column X_T represents the vector of mn observations given by

$$X_T = (X_{11}, \cdots X_{1n}, X_{21}, \cdots X_{2n}, \cdots, X_{m1}, \cdots, Xmn)'.$$

Then it is clear that $X_T \sim N_T(\mu V, \Sigma)$, where V is a column vector of 1's of order $T \times 1$ and Σ is the cavariance matrix (order $T \times T$) of X_T. From the independence of $X_i's$, obviously, Σ is a block diagonal matrix such that $\Sigma = diag(\Omega)$, where Ω is the covariance matrix (order $n \times n$) of any single series X_i. Since the autocovariance function of any single series is $\gamma_k = \frac{\sigma^2 \phi^k}{1-\phi^2}$, the symmetric $n \times n$ matrix Ω is given by

$$\Omega = \frac{\sigma^2}{1-\phi^2} \begin{pmatrix} 1 & \phi & \phi^2 & \dots & \phi^{n-1} \\ \phi & 1 & \phi & \dots & \phi^{n-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \phi^{n-1} & \dots & \dots & \dots & 1 \end{pmatrix}.$$

The joint probability density function of X_T is

$$f(X_T, \Sigma) = (2\pi)^{\frac{-T}{2}} |\Sigma|^{\frac{-1}{2}} exp\{\frac{-1}{2}(X_T - \mu V)'\Sigma^{-1}(X_T - \mu V)\} \tag{2.10}$$

$$= (1-\phi^2)^{mn/2}(1-\pi^2)^{-m/2} exp\{\frac{-1}{2\sigma^2}\sum_{j=1}^{m}[(X_{j1}-\mu)^2+(1+\phi^2)\sum_{t=2}^{n}(X_{jt}-\mu-\phi(X_{j,t-1}-\mu))^2]\} \tag{2.11}$$

Equation (2.11) is equivalent to

$$\frac{(1-\phi^2)^{m/2}}{(2\pi\sigma^2)^{mn/2}} exp\{\frac{-1}{2\sigma^2}\sum_{j=1}^{m}[(X_{j1}-\mu)^2 + (1+\phi^2)\sum_{t=2}^{n-1}(X_{jt}-\mu)^2 + (X_{j,n}-\mu)^2$$

$$- 2\phi \sum_{t=1}^{n-1}(X_{jt}-\mu)(X_{j,t+1}-\mu)]\}.$$

To estimate the parameters ϕ, μ and σ^2, one needs a suitable optimization algorithm to maximize the likelihood function given in (2.11) (or the corresponding log likelihood function). As we have the covariance matrix $\Sigma = diag(\Omega)$ in terms of the parameters ϕ and σ^2, the exact mle's can easily be obtained by choosing an appropriate set of starting up values for the optimization algorithm. Denote the corresponding vector of the estimates by $\hat{\delta}_2$, where $\hat{\delta}_2 = (\hat{\phi}_2, \hat{\mu}_2, \hat{\sigma}_2^2)'$.

Next section we compare the finite sample properties of $\hat{\delta}_1$ and $\hat{\delta}_2$ via a simulation study with corresponding asymptotic results.

3 Finite Sample Comparison of $\hat{\delta}_1$ and $\hat{\delta}_2$

The properties of $\hat{\delta}_1$, and $\hat{\delta}_2$ are are very similar to each other for large values of m, especially the bias and the mean square error (mse). However, there are slight differences for small values of m. It can be seen that the asymptotic covariance matrix of $\hat{\delta}_1$ based on (2.1) is

$$var(\hat{\delta}_1) = \frac{1}{m(n-1)} \begin{pmatrix} 1-\phi^2 & 0 & 0 \\ 0 & \frac{\sigma^2}{(1-\phi)^2} & 0 \\ 0 & 0 & 2\sigma^4 \end{pmatrix}.$$

The corresponding matrix for the exact mle of δ_2 based on (2.11) is

$$var(\hat{\delta}_2) = \frac{1}{ac-d^2} \begin{pmatrix} c & 0 & -d \\ 0 & \frac{ac-d^2}{b} & 0 \\ -d & 0 & a \end{pmatrix},$$

where

$$a = \frac{m[\phi^2(3-n)+n-1]}{(1-\phi^2)^2},\ b = \frac{m(1-\phi)[n-(n-2)\phi]}{\sigma^2}, c = \frac{mn}{2\sigma^4},\ d = \frac{m\phi}{\sigma^2(1-\phi^2)}.$$

From this matrix, it is clear that asymptotically (for large m) $\hat{\phi}$ is normal with mean ϕ and variance $\frac{c}{ac-d^2}$.

For example when $m = 100$ and $n = 5$ ($\phi = .8,\ \mu = 4,\ \sigma^2 = 2$), we have

$$var(\hat{\delta}_1)_t = \begin{pmatrix} 0.0009 & 00.0000 & 0.0000 \\ 0.0000 & 0.0278 & 0.0000 \\ 0.0000 & 0.0000 & 0.0800 \end{pmatrix}$$

$$var(\hat{\delta}_2)_t = \begin{pmatrix} 0.0005 & 00.0000 & -0.0001 \\ 0.0000 & 0.0385 & 0.0000 \\ -0.0001 & 0.0000 & 0.0080 \end{pmatrix}$$

where t stands for corresponding theoretical values.

Under the null hypothesis of no serial correlation, ie. $\phi = 0$, the null distribution of $\hat{\phi}$ is asymptotically normal with mean 0 and variance $\frac{1}{m(n-1)}$.

Note:- Using the estimator of ϕ as

$$\hat{\phi} = \frac{\sum_{i=1}^{m}\sum_{t=1}^{n-1}(X_{it}-\bar{X}_i)(X_{i,t+1}-\bar{X}_i)}{\sum_{i=1}^{m}\sum_{t=1}^{n}(X_{it}-\bar{X}_i)^2}, \tag{3.1}$$

the corresponding null distribution (for large m) of $\hat{\phi}$ when $n = 5$ is normal with mean $\frac{-1}{5}$ and variance $\frac{27}{200m}$. (See Cox and Solomon (1988), p.147). Although this has a smaller variance than the previous one (i.e.,$\frac{1}{4m}$), it has a considerable bias.

Next section we consider a simulation study.

4 A Simulation Study

We first generate a sample of 100 from (1.3) for a given set of parameters ϕ, μ, and σ^2 using Splus and repeat this m times. Now pick the last five columns of this matrix (of order $m \times 100$) and take (a matrix of order $m \times 5$) as our sample. Using this sample and equations (2.6), (2.8) and (2.9) compute $\hat{\delta}_1$ based on the conditional argument. The exact likelihood estimates of parameters (or $\hat{\delta}_2$) are obtained by numerically maximizing (2.11) using the Newton-Raphson method. For $n = 5$, we repeat the simulation and estimation using both ML procedures for different values m and k. At the end of the each estimation, we compute the mean and variance of $\hat{\delta}_1$ and $\hat{\delta}_2$. Further the *bias* and *mean square error (mse)* of $\hat{\delta}_1$ and $\hat{\delta}_2$ are obtained for comparison. Let $\hat{\delta}_{i.j}$ stands for the j^{th} estimate of the vector $\hat{\delta}_i$. ; i=1,2.

Then

$$bias_i = \frac{1}{k}\sum_{j=1}^{k}(\hat{\delta}_{i.j}-\delta_i), \tag{4.1}$$

and

$$mse_i = \frac{1}{k}\sum_{j=1}^{k}(\hat{\delta}_{i.j}-\delta_i)^2, \tag{4.2}$$

We tabulate these results in Appendix for various values of m and k.

Each table has four parts consisting

- the true values of δ and means of estimated values of $\hat{\delta}_1$ and $\hat{\delta}_2$.
- the variances of estimated values of $\hat{\delta}_1$ and $\hat{\delta}_2$.
- summarize the bias of the estimates of $\hat{\delta}_1$ and $\hat{\delta}_2$.
- mean square errors of the estimates of $\hat{\delta}_1$ and $\hat{\delta}_2$

For given $\phi = 0.8, \mu = 4$ and $\sigma^2 = 2$, We compute the covariance matrices of $\hat{\delta}_1$ and $\hat{\delta}_2$ based on our simulations for $m = 100$ and $n = 5$. we denote these matrices by $var(\hat{\delta}_1)_s$ and $var(\hat{\delta}_2)_s$ for convenience (s stands for simulation).

$$var(\hat{\delta}_1)_s = \begin{pmatrix} 0.0009 & 00.0012 & 0.0000 \\ 0.0012 & 0.1367 & 0.0000 \\ 0.0000 & 0.0000 & 0.0180 \end{pmatrix}$$

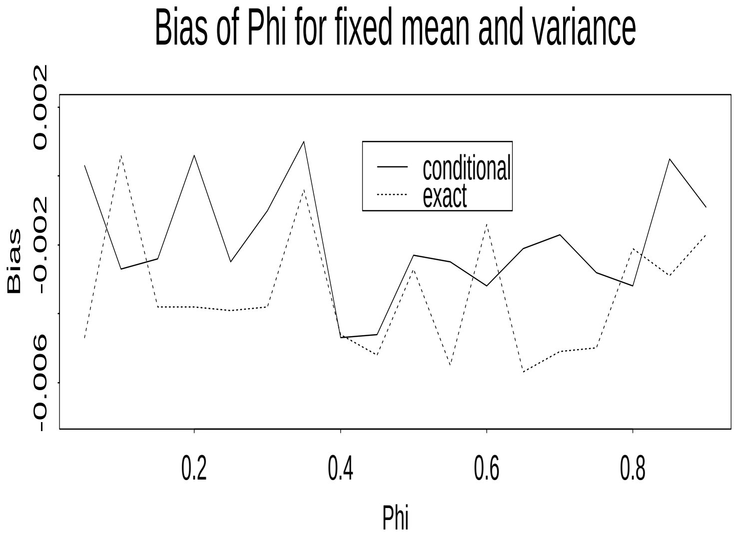
$$var(\hat{\delta}_2)_s = \begin{pmatrix} 0.0005 & 00.0000 & -0.0009 \\ 0.0000 & 0.0391 & 0.0000 \\ -0.0009 & 0.0000 & 0.0160 \end{pmatrix}$$

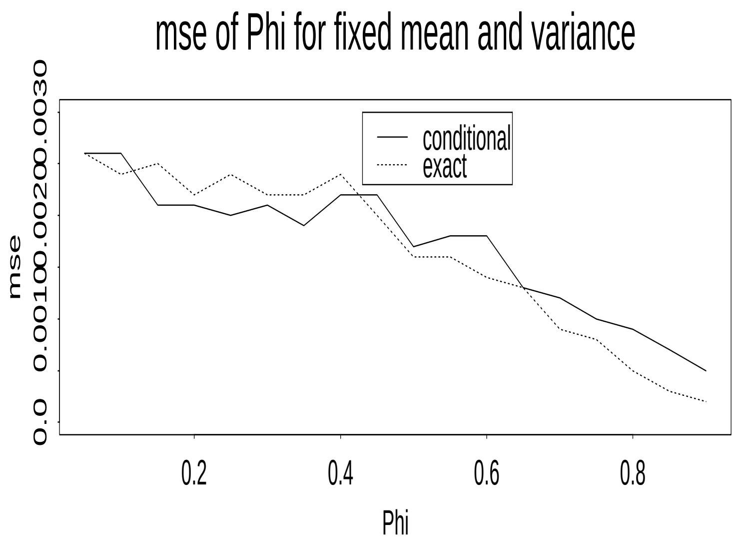
Notice that in each case (conditional mle and exact mle) $var(\hat{\delta}_i)_t \approx var(\hat{\delta}_i)_s$ for i=1,2.

Below we give a graphical representations for the bias and the mse of $\hat{\phi}$ in both conditional and exact cases.

Comparison of the bias and the mse of $\hat{\phi}$

$(\mu = 5 \ and \ \sigma^2 = 1, m = 100, k = 300)$





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APPENDIX

TABLE 1 : m=100 k=300

Table 1.1 -Simulated means of $\hat{\delta}_1$ and $\hat{\delta}_2$

δ	$\hat{\delta}_1$,Conditional mle	$\hat{\delta}_2$,Exact mle
(0.8,4,1.0)	(0.7986,4.0019,0.9959)	(0.7964,3.9897,0.9991)
(0.7,5,1.5)	(0.6973,5.0137,1.5013)	(0.6973,5.0049,1.4890)
(0.6,6,2.0)	(0.5938,6.0090,1.9928)	(0.5967,6.0030,1.9920)
(0.5,5,1.0)	(0.4927,4.9930,1.0040)	(0.4961,5.0012,1.0006)
(0.4,8,5.0)	(0.3975,7.9734,5.0216)	(0.3967,8.0039,4.9618)
(0.3,10,7.0)	(0.2952,10.0001,6.9853)	(0.2973,10.0086,6.9624)
(0.2,4,3.0)	(0.1952,4.0052,3.0048)	(0.1999,4.0008,2.9898)
(0.1,5,2.0)	(0.1007,4.9965,1.9879)	(0.1006,5.0013,1.9888)

Table 1.2 -Simulated variances of $\hat{\delta}_1$ and $\hat{\delta}_2$

δ	$\hat{\delta}_1$,Conditional mle	$\hat{\delta}_2$,Exact mle
(0.8,4,1.0)	(0.0009,0.0613,0.0040)	(0.0005,0.0196,0.0042)
(0.7,5,1.5)	(0.0012,0.0489,0.0104)	(0.0010,0.0173,0.0107)
(0.6,6,2.0)	(0.0018,0.0282,0.0201)	(0.0013,0.0168,0.0166)
(0.5,5,1.0)	(0.0021,0.0104,0.0051)	(0.0018,0.0056,0.0045)
(0.4,8,5.0)	(0.0021,0.0400,0.1308)	(0.0019,0.0239,0.0901)
(0.3,10,7.0)	(0.0022,0.0394,0.2150)	(0.0020,0.0236,0.1932)
(0.2,4,3.0)	(0.0025,0.0124,0.0471)	(0.0022,0.0091,0.0316)
(0.1,5,2.0)	(0.0023,0.0067,0.0199)	(0.0023,0.0048,0.0143)

Table 1.3 -Simulated bias of $\hat{\delta}_1$ and $\hat{\delta}_2$

δ	$\hat{\delta}_1$,Conditional mle	$\hat{\delta}_2$,Exact mle
(0.8,4,1.0)	(-0.0014,0.0019,-0.0041)	(-0.0036,-0.0103,-0.0009)
(0.7,5,1.5)	(-0.0027,0.0137,0.0013)	(-0.0027,0.0049,-0.0110)
(0.6,6,2.0)	(-0.0062,0.0090,-0.0072)	(-0.0033,0.0030,-0.0080)
(0.5,5,1.0)	(-0.0073,-0.0070,0.0040)	(-0.0039,0.0012,0.0006)
(0.4,8,5.0)	(-0.0025,-0.0266,0.0216)	(-0.0033,0.0039,-0.0382)
(0.3,10,7.0)	(-0.0048,0.0001,-0.0147)	(-0.0027,0.0086,-0.0376)
(0.2,4,3.0)	(-0.0048,0.0052,0.0048)	(-0.0001,0.0008,-0.0102)
(0.1,5,2.0)	(0.0007,-0.0035,0.0121)	(0.0006,0.0013,-0.0112)

Table 1.4 -Simulated mse of $\hat{\delta}_1$ and $\hat{\delta}_2$

δ	$\hat{\delta}_1$,Conditional mle	$\hat{\delta}_2$,Exact mle
(0.8,4,1.0)	(0.0009,0.0613,0.0041)	(0.0005,0.0197,0.0042)
(0.7,5,1.5)	(0.0012,0.0491,0.0104)	(0.0010,0.0173,0.0108)
(0.6,6,2.0)	(0.0018,0.0283,0.0201)	(0.0013,0.0168,0.0166)
(0.5,5,1.0)	(0.0021,0.0104,0.0051)	(0.0018,0.0056,0.0045)
(0.4,8,5.0)	(0.0021,0.0407,0.1312)	(0.0019,0.0240,0.0915)
(0.3,10,7.0)	(0.0022,0.0394,0.2152)	(0.0020,0.0237,0.1946)
(0.2,4,3.0)	(0.0025,0.0124,0.0471)	(0.0022,0.0091,0.0317)
(0.1,5,2.0)	(0.0023,0.0067,0.0201)	(0.0023,0.0048,0.0145)

TABLE 2 : m=500 k=500

Table 2.1 -Simulated means of $\hat{\delta}_1$ and $\hat{\delta}_2$

δ	$\hat{\delta}_1$,Conditional mle	$\hat{\delta}_2$,Exact mle
(0.8,4,1.0)	(0.7988,3.9959,0.9979)	(0.7995,4.0003,0.9983)
(0.7,5,1.5)	(0.6988,4.9975,1.4959)	(0.6999,4.9994,1.4986)
(0.6,6,2.0)	(0.5996,6.0032,1.9988)	(0.5985,6.0003,2.0019)
(0.5,5,1.0)	(0.5009,4.9973,1.0009)	(0.4983,5.0007,0.9999)
(0.4,8,5.0)	(0.4012,7.9934,4.9918)	(0.4000,8.0058,4.9963)
(0.3,10,7.0)	(0.2986,9.9990,7.0031)	(0.2995,10.0005,7.0058)
(0.2,4,3.0)	(0.1992,3.9975,2.9954)	(0.2006,4.0014,3.0006)
(0.1,5,2.0)	(0.1003,5.0002,2.0029)	(0.0986,5.0024,1.9965)

Table 2.2 -Simulated variances of $\hat{\delta}_1$ and $\hat{\delta}_2$

δ	$\hat{\delta}_1$,Conditional mle	$\hat{\delta}_2$,Exact mle
(0.8,4,1.0)	(0.0002,0.0135,0.0010)	(0.0001,0.0040,0.0009)
(0.7,5,1.5)	(0.0003,0.0077,0.0021)	(0.0002,0.0035,0.0021)
(0.6,6,2.0)	(0.0003,0.0069,0.0042)	(0.0002,0.0033,0.0034)
(0.5,5,1.0)	(0.0004,0.0019,0.0011)	(0.0003,0.0010,0.0008)
(0.4,8,5.0)	(0.0004,0.0065,0.0274)	(0.0004,0.0037,0.0194)
(0.3,10,7.0)	(0.0004,0.0075,0.0481)	(0.0004,0.0049,0.0411)
(0.2,4,3.0)	(0.0005,0.0023,0.0090)	(0.0005,0.0018,0.0065)
(0.1,5,2.0)	(0.0005,0.0013,0.0037)	(0.0004,0.0010,0.0032)

Table 2.3 -Simulated bias of $\hat{\delta}_1$ and $\hat{\delta}_2$

δ	$\hat{\delta}_1$,Conditional mle	$\hat{\delta}_2$,Exact mle
(0.8,4,1.0)	(-0.0012,-0.0041,-0.0021)	(-0.0005,0.0003,-0.0017)
(0.7,5,1.5)	(-0.0012,-0.0025,-0.0041)	(-0.0001,-0.0006,-0.0014)
(0.6,6,2.0)	(-0.0004,0.0032,-0.0012)	(-0.0015,0.0003,0.0019)
(0.5,5,1.0)	(0.0009,-0.0027,0.0009)	(-0.0017,0.0007,-0.0001)
(0.4,8,5.0)	(0.0012,-0.0066,-0.0082)	(0.0000,0.0058,-0.0037)
(0.3,10,7.0)	(-0.0014,-0.0010,0.0031)	(-0.0005,0.0005,0.0058)
(0.2,4,3.0)	(-0.0008,-0.0025,-0.0046)	(0.0006,0.00014,0.0006)
(0.1,5,2.0)	(0.0003,0.0002,0.0029)	(-0.0014,0.0024,-0.0035)

Table 2.4 -Simulated mse of $\hat{\delta}_1$ and $\hat{\delta}_2$

δ	$\hat{\delta}_1$,Conditional mle	$\hat{\delta}_2$,Exact mle
(0.8,4,1.0)	(0.0002,0.0135,0.0010)	(0.0001,0.0040,0.0009)
(0.7,5,1.5)	(0.0003,0.0077,0.0021)	(0.0002,0.0035,0.0021)
(0.6,6,2.0)	(0.0003,0.0069,0.0042)	(0.0003,0.0033,0.0034)
(0.5,5,1.0)	(0.0004,0.0019,0.0011)	(0.0003,0.0010,0.0008)
(0.4,8,5.0)	(0.0004,0.0066,0.0275)	(0.0004,0.0037,0.0194)
(0.3,10,7.0)	(0.0004,0.0075,0.0481)	(0.0004,0.0049,0.0411)
(0.2,4,3.0)	(0.0005,0.0023,0.0090)	(0.0005,0.0018,0.0065)
(0.1,5,2.0)	(0.0005,0.0013,0.0037)	(0.0004,0.0010,0.0033)

TABLE 3 : m=1000 k=1000

Table 3.1 -Simulated means of $\hat{\delta}_1$ and $\hat{\delta}_2$

δ	$\hat{\delta}_1$,Conditional mle	$\hat{\delta}_2$,Exact mle
(0.8,4,1.0)	(0.7998,40030,0.9985)	(0.7998,4.0005,0.9986)
(0.7,5,1.5)	(0.7000,5.0020,1.4981)	(0.6992,5.0007,1.4996)
(0.6,6,2.0)	(0.5995,5.9986,1.9993)	(0.5998,5.9980,1.9986)
(0.5,5,1.0)	(0.4995,4.9999,0.9994)	(0.4998,5.0004,1.0000)
(0.4,8,5.0)	(0.3993,7.9997,4.9981)	(0.4001,7.9992,5.0006)
(0.3,10,7.0)	(0.3003,9.9999,6.9903)	(0.3000,9.9979,6.9999)
(0.2,4,3.0)	(0.1999,3.9995,2.9972)	(0.1993,4.0003,2.9961)
(0.1,5,2.0)	(0.0992,5.0000,1.9980)	(0.1003,4.9999,1.9997)

Table 3.2 -Simulated variances of $\hat{\delta}_1$ and $\hat{\delta}_2$

δ	$\hat{\delta}_1$,Conditional mle	$\hat{\delta}_2$,Exact mle
(0.8,4,1.0)	(0.0001,0.0063,0.0005)	(0.0001,0.0019,0.0004)
(0.7,5,1.5)	(0.0001,0.0043,0.0011)	(0.0001,0.0017,0.0009)
(0.6,6,2.0)	(0.0002,0.0032,0.0020)	(0.0001,0.0016,0.0016)
(0.5,5,1.0)	(0.0002,0.0010,0.0005)	(0.0002,0.0006,0.0004)
(0.4,8,5.0)	(0.0002,0.0034,0.0127)	(0.0002,0.0021,0.0104)
(0.3,10,7.0)	(0.0003,0.0035,0.0234)	(0.0002,0.0023,0.0200)
(0.2,4,3.0)	(0.0002,0.0011,0.0046)	(0.0002,0.0008,0.0036)
(0.1,5,2.0)	(0.0002,0.0006,0.0020)	(0.0003,0.0005,0.0018)

Table 3.3 -Simulated bias of $\hat{\delta}_1$ and $\hat{\delta}_2$

δ	$\hat{\delta}_1$,Conditional mle	$\hat{\delta}_2$,Exact mle
(0.8,4,1.0)	(-0.0002,0.0030,-0.0015)	(-0.0002,0.0005,-0.0014)
(0.7,5,1.5)	(0.0000,0.0020,-0.0019)	(-0.0008,0.0007,-0.0004)
(0.6,6,2.0)	(-0.0005,-0.0014,-0.0007)	(-0.0002,-0.0020,-0.0014)
(0.5,5,1.0)	(-0.0005,-0.0001,-0.0006)	(-0.0002,0.0004,0.0000)
(0.4,8,5.0)	(-0.0007,-0.0003,-0.0019)	(0.0001,-0.0008,0.0006)
(0.3,10,7.0)	(0.0003,-0.0001,-0.0092)	(0.0000,-0.0021,0.0001)
(0.2,4,3.0)	(-0.0001,-0.0005,-0.0028)	(-0.0007,0.00003,-0.0039)
(0.1,5,2.0)	(-0.0008,0.0000,-0.0020)	(0.0003,-0.0001,-0.0003)

Table 3.4 -Simulated mse of $\hat{\delta}_1$ and $\hat{\delta}_2$

δ	$\hat{\delta}_1$,Conditional mle	$\hat{\delta}_2$,Exact mle
(0.8,4,1.0)	(0.0001,0.0063,0.0005)	(0.0001,0.0019,0.0005)
(0.7,5,1.5)	(0.0001,0.0043,0.0011)	(0.0001,0.0017,0.0009)
(0.6,6,2.0)	(0.0002,0.0032,0.0020)	(0.0001,0.0016,0.0016)
(0.5,5,1.0)	(0.0002,0.0010,0.0005)	(0.0002,0.0006,0.0004)
(0.4,8,5.0)	(0.0002,0.0034,0.0127)	(0.0002,0.0021,0.0104)
(0.3,10,7.0)	(0.0003,0.0035,0.0235)	(0.0002,0.0023,0.0200)
(0.2,4,3.0)	(0.0002,0.0011,0.0046)	(0.0002,0.0008,0.0036)
(0.1,5,2.0)	(0.0002,0.0006,0.0020)	(0.0003,0.0005,0.0018)