LINEAR ESTIMATION OF STANDARD DEVIATION OF LOGISTIC DISTRIBUTION: THEORY AND ALGORITHM

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Abstract: The paper presents a theoretical method based on order statistics and a FORTRAN program for computing the variance and relative efficiencies of the standard deviation of the logistic population with respect to the Cramer-Rao lower variance bound and the best linear unbiased estimators (BLUE's) when the mean is unknown. A method based on a pair of single spacing and the 'zero-one' weights rather than the optimum weights are used. A comparison of an estimator based on four order statistics with the traditional estimators is considered.

Key words: Order Statistics, Logistic Distribution, Parameter Estimation, Fortran, Relative Efficiencies.

2000 Math. Subject Classification: 62 G30, 62 F10, 65 C60

INTRODUCTION

Let \( X_{1:n} \leq \ldots \leq X_{n:n} \) denote the order statistics from a random sample of size \( n \) from the logistic distribution whose cumulative distribution function is

\[
F(x; \mu, \sigma) = \left[ 1 + \exp\left\{ - (x - \mu) / (\tau \sigma) \right\} \right]^{-1},
\]

where \(-\infty < x < \infty, \ -\infty < \mu < \infty, \ \sigma > 0 \) and \( \tau = \sqrt{3}/\pi \). The distribution is absolutely continuous, symmetric about the location parameter, the mean \( \mu \), and with scale parameter \( \sigma \). Dixon [4] introduced the notion of using 'zero-one' weights rather than the optimum weights and noted that high efficiencies are achievable. This work has since then been extended by many authors, notably, Raghunandanan and Srinivasan [11] for the logistic distribution and Wang Cheng-guan [12] for the normal distribution. Raghunandanan and Srinivasan [11] constructed simplified estimators of the location and scale parameters for complete and symmetrically censored samples for sample sizes \( 4 \leq n \leq 20 \).

The principal object of this paper is to develop, based on the work of Wang Cheng-guan [12], a simplified linear estimator of the scale parameter of the population when its location parameter is unknown into a FORTRAN computer program. The desired procedure should be applicable to both censored and uncensored samples and follows the work of Keats et al.[8]. This estimator will be shown to have bias less than 0.5% for \( n \geq 5 \) and this bias reduces rapidly to zero as \( n \) increases. The asymptotic variance and asymptotic efficiency are also given. Furthermore, it is shown that the estimator has high relative efficiency with respect to the 'zero-one' linear estimator of the scale parameter (\( eff_{0-1} \)) for \( n \) between 5 and 20 and we note that this procedure is developed as a compromise between lack of efficiency and quickness and ease of computation.

ESTIMATOR OF STANDARD DEVIATION

The method discussed here is based on a pair of single spacing and the expectation of
the sum of consecutive order statistics in a sample of size \( n \). Let a spacing \( c \) be defined in relation to the rank \( i \) of order statistics \( X \) and sample size \( n \) as

\[
i = \left[ nc + 0.5 \right]
\]

where \([x]\) denotes the greatest integer \( \leq x \) and let

\[
G(u) = F^{-1}(u) = \tau \log(u/(1-u)), \quad 0 < u < 1
\]

be the inverse function, then this expectation is given by

\[
E(X_{i:n} + X_{i+1:n}) = 2 \left( G(c^*) + G'(c^*) \left( \frac{i}{n} - c^* \right) \right) + \frac{G''(c^*)}{12n^2} + O\left( \frac{1}{n^3} \right), \quad 1 \leq i \leq n-1.
\]

In this paper, we have used \( c^* = \frac{i}{n} \) in all the calculations. Let \( R_i \) denote the \( i \)-th quasi-range

\[
R_i = X_{n-i:n} - X_{i+1:n}.
\]

### Table 1: Ranks, Bias, Variances and Efficiencies for various \( n=5(1)20(5)50, \infty \)

<table>
<thead>
<tr>
<th>n</th>
<th>i</th>
<th>i+1</th>
<th>n-i</th>
<th>n-i+1</th>
<th>BIAS</th>
<th>( V(\hat{\sigma}_1)/\sigma^2 )</th>
<th>( V(\hat{\sigma})/\sigma^2 )</th>
<th>eff_1</th>
<th>eff_2</th>
<th>eff_{0-1}</th>
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<tbody>
<tr>
<td>5</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>5</td>
<td>0.0002</td>
<td>0.1720</td>
<td>0.1704</td>
<td>0.813</td>
<td>0.991</td>
<td>1.000</td>
</tr>
<tr>
<td>6</td>
<td>1</td>
<td>2</td>
<td>5</td>
<td>6</td>
<td>0.0001</td>
<td>0.1377</td>
<td>0.1429</td>
<td>0.847</td>
<td>0.995</td>
<td>0.999</td>
</tr>
<tr>
<td>7</td>
<td>1</td>
<td>2</td>
<td>6</td>
<td>7</td>
<td>0.0000</td>
<td>0.1160</td>
<td>0.1232</td>
<td>0.861</td>
<td>0.987</td>
<td>0.999</td>
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<tr>
<td>8</td>
<td>1</td>
<td>2</td>
<td>7</td>
<td>8</td>
<td>0.0000</td>
<td>0.1011</td>
<td>0.1052</td>
<td>0.865</td>
<td>0.973</td>
<td>0.999</td>
</tr>
<tr>
<td>9</td>
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<td>2</td>
<td>8</td>
<td>9</td>
<td>0.0000</td>
<td>0.0903</td>
<td>0.0918</td>
<td>0.861</td>
<td>0.956</td>
<td>0.978</td>
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<tr>
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<td>2</td>
<td>9</td>
<td>10</td>
<td>0.0000</td>
<td>0.0820</td>
<td>0.0850</td>
<td>0.853</td>
<td>0.937</td>
<td>0.958</td>
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<tr>
<td>11</td>
<td>1</td>
<td>2</td>
<td>10</td>
<td>11</td>
<td>0.0000</td>
<td>0.0754</td>
<td>0.0788</td>
<td>0.843</td>
<td>0.917</td>
<td>0.943</td>
</tr>
<tr>
<td>12</td>
<td>1</td>
<td>2</td>
<td>11</td>
<td>12</td>
<td>0.0000</td>
<td>0.0701</td>
<td>0.0711</td>
<td>0.831</td>
<td>0.898</td>
<td>0.920</td>
</tr>
<tr>
<td>13</td>
<td>1</td>
<td>2</td>
<td>12</td>
<td>13</td>
<td>0.0000</td>
<td>0.0658</td>
<td>0.0648</td>
<td>0.818</td>
<td>0.878</td>
<td>0.900</td>
</tr>
<tr>
<td>14</td>
<td>1</td>
<td>2</td>
<td>13</td>
<td>14</td>
<td>0.0000</td>
<td>0.0621</td>
<td>0.0634</td>
<td>0.805</td>
<td>0.860</td>
<td>0.881</td>
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<tr>
<td>15</td>
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<td>3</td>
<td>13</td>
<td>14</td>
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<td>0.0600</td>
<td>0.0584</td>
<td>0.777</td>
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</tr>
<tr>
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<td>0.782</td>
<td>0.828</td>
<td>0.843</td>
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<tr>
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<td>3</td>
<td>15</td>
<td>16</td>
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<td>0.0527</td>
<td>0.784</td>
<td>0.827</td>
<td>0.830</td>
</tr>
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<td>3</td>
<td>16</td>
<td>17</td>
<td>0.0000</td>
<td>0.0495</td>
<td>0.0491</td>
<td>0.784</td>
<td>0.825</td>
<td>0.827</td>
</tr>
<tr>
<td>19</td>
<td>2</td>
<td>3</td>
<td>17</td>
<td>18</td>
<td>0.0000</td>
<td>0.0470</td>
<td>0.0460</td>
<td>0.783</td>
<td>0.822</td>
<td>0.826</td>
</tr>
<tr>
<td>20</td>
<td>2</td>
<td>3</td>
<td>18</td>
<td>19</td>
<td>0.0000</td>
<td>0.0448</td>
<td>0.0439</td>
<td>0.781</td>
<td>0.818</td>
<td>0.818</td>
</tr>
<tr>
<td>25</td>
<td>3</td>
<td>4</td>
<td>22</td>
<td>23</td>
<td>0.0000</td>
<td>0.0372</td>
<td>0.0352</td>
<td>0.751</td>
<td>0.779</td>
<td></td>
</tr>
<tr>
<td>30</td>
<td>3</td>
<td>4</td>
<td>27</td>
<td>28</td>
<td>0.0000</td>
<td>0.0310</td>
<td>0.0310</td>
<td>0.752</td>
<td>0.775</td>
<td></td>
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<tr>
<td>35</td>
<td>4</td>
<td>5</td>
<td>31</td>
<td>32</td>
<td>0.0000</td>
<td>0.0271</td>
<td>0.0271</td>
<td>0.736</td>
<td>0.755</td>
<td></td>
</tr>
<tr>
<td>40</td>
<td>4</td>
<td>5</td>
<td>36</td>
<td>37</td>
<td>0.0000</td>
<td>0.0238</td>
<td>0.0238</td>
<td>0.736</td>
<td>0.752</td>
<td></td>
</tr>
<tr>
<td>45</td>
<td>5</td>
<td>6</td>
<td>40</td>
<td>41</td>
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<td>0.0214</td>
<td>0.0214</td>
<td>0.726</td>
<td>0.741</td>
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<tr>
<td>50</td>
<td>5</td>
<td>6</td>
<td>45</td>
<td>46</td>
<td>0.0000</td>
<td>0.0193</td>
<td>0.0193</td>
<td>0.726</td>
<td>0.737</td>
<td></td>
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<tr>
<td>( \infty )</td>
<td>0.0000</td>
<td>1.0227/n</td>
<td>0.684</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Then based on Equations (3) and (4), we propose to use

\[
\hat{\sigma}_1 = \frac{(R_i + R_{i-1})}{2E_{i,i+1:n}},
\]

where \( E_{i,i+1:n} = -E_{n-i,n-i+1:n} \) due to symmetry.
\[ E_{i,i+1n} = 2 \left\{ \frac{G(1-c^*) - G^*(c^*)}{12n^2} \right\}, \quad (6) \]

\[ = 2 \pi \left\{ \log \left( \frac{n-i}{i} \right) + \frac{n(n-2i)}{12i^2 (n-i)^2} \right\} \quad (7) \]
as the estimator of the scale parameter \( \sigma \). The methods for estimating the location parameter, \( \mu \), of logistic distribution is not covered in this paper and readers are referred to Weke et al. [13].

**EXAMPLE**

A single spacing value of 0.10293 has been established from Gupta and Gnanadesikan [5] and is used here in all computations. Let \( V(\hat{\sigma}_1) \) (given in the program as VAR) and \( V(\hat{\sigma}) \) be the variances of \( \hat{\sigma}_1 \) and that of the estimator based on four optimum order statistics, respectively, then we have

\[ \text{eff}_1 = \frac{9}{n(3 + \pi^2)} V(\hat{\sigma}_1) \]

and

\[ \text{eff}_2 = \frac{\text{Var}[\text{BLUE}(\sigma)]}{V(\hat{\sigma}_1)} = \frac{\text{Var}_{\text{opt}}}{\text{VAR}} \]
as the relative efficiencies of \( \hat{\sigma}_1 \) with respect to the Cramer-Rao lower bound for the variance of an unbiased estimator of \( \sigma \) and the variance of BLUE of \( \sigma \), respectively. The efficiencies \( \text{eff}_1 \) and \( \text{eff}_2 \) are given in the program as AEF and R.E., respectively. Table 1 gives the ranks of order statistics, the bias and variance of the estimator \( \hat{\sigma}_1 \) and the efficiencies relative to Cramer-Rao lower variance bound, the BLUE and the ‘zero-one’ linear estimators of scale parameter for various values of \( n \). For illustration, we consider the case when \( n = 20 \) and by using Equations (1) and (5) we have the numerator as \( 2(1.37563 + 1.06933) = 4.88992 \) and denominator as 4.89096 when \( i = 2 \). This leads to a bias of 0.00002. This method uses four order statistics instead of the usual two order statistics. It should be mentioned that bias equal to zero does not imply that the estimate in Equation (7) coincides with the expected value of the two consecutive order statistics; it simply means that the error is less than \( 5 \times 10^{-5} \).

In conclusion, we notice that \( \text{eff}_1 \leq \text{eff}_2 \leq \text{eff}_{0,1} \). A comparison with an estimator, abbreviated by \( \hat{\theta} \), based on four optimum order statistics given by Chan et al. [3] reveals that the variances of the estimator in Equation (5) is smaller than \( V(\hat{\theta}) \) for most values of \( n \) within the range \( 5 \leq n \leq 25 \). The variances required to calculate \( \text{eff}_{0,1} \) values are available in Balakrishnan and Cohen [1] page 255 for \( 2 \leq n \leq 20 \). All computations were performed in double precision by FORTRAN 77 language programs to produce results for values up to \( n = 100 \) but to save space we have not included all these cases in our table. The computer programs for the calculation of the means, product moments, variances and covariances of the logistic order statistics and their respective data files necessary for the construction of Table 1 are too huge to be reproduced here. However, their values are found to coincide with those values in Birnbaum and Dudman [2], Gupta et al. [6], and Harter and Balakrishnan [7]. The procedure discussed in this paper produces reasonably good results, quick and easy computations. Finally, it is noted that no reference is made to pre-existing tables for the computation of the estimator and its variance and efficiencies.

**PROGRAM DESCRIPTION**

The FORTRAN program computes the variance of the estimator given in Equation (5), its efficiencies relative to the Cramer-Rao lower variance bound and the BLUE's and gives the variance-covariance matrix of
the logistic order statistics as an output. All the features are clearly illustrated in the program, notably,

INPUT FILES: LOGIS1.EXP and LOGIS1.COV are DATA FILES for the expected and covariance values for order statistics. These have been calculated by using a different program. However, similar values can be found in Chan et al. [3]. Input $K K_2 = 2$; $N_0 N_6 = 5, n$ and $IJ(I) = i, CF1(I) = i + 1$ according to Table 1.

DISPLAY: The lower-triangular matrix of covariance. We have used the Cholesky's inverse matrix and Lloyd's optimal methods

EFFICIENCIES: $AEF = eff_1$ and $R.E. = V_{opt}/VAR = eff_2$.

SUBROUTINES: MINV - Employs Cholesky's method to compute inverse of the matrix.

VARML - Output a lower triangular matrix row by row. The lower triangle is stored in a 1-dimensional array.

VLTV - Output a vector with a title.

REFERENCES
Censored Data". *Journal of Quality Technology* 29, 105-110.


**PROGRAM LISTING**

```c
C VARIL.FOR   by  PATRICK WEKE  at HIT/UoN
C COMPUTE THE VARIANCES of The
   ESTIMATES for THE SCALE
C LOGISTIC POPULATION.
C   I=[c*(N+1-2*a)+a]  , a=0.5
INTEGER N,N0,N6,M,I,J ,I2,J2,L,K, M1, M2
   ,K1,K2,IV,KK,KK2
INTEGER N1(100),N2(100), N3(100),
   N4(100), N5(100),I,J1(100),S(100)
REAL RN,BB1,BB2,RN1(100),RN2(100) ,
   RN3(100), RN4(100),RN5(100)
REAL a,c,NUM,DE,SC(10),CF(100),
   CF1(100), W(100),RX(820),R(100)
REAL VAR,AEF,RY(820),COP(100),R1(100)
CHARACTER*8 FNAME
C   M2:     NUMBER of ORDER STATISTICS
   USED, M1=(M2+1)/2
C   CF(K):  COEFFICIENTS of PAIR of
   CONSECUTIVE ORDER STATISTICS
C   CF1(K): COEFFICIENTS of EVERY
   ORDER STATISTICS
C   W(K):  WEIGHT of EVERY ORDER
   STATISTICS
C   N:   SIZE, STARTING:N0, FINISH:N6
C   R(L) :  THE SUM of EXPECTATION of
   ORDER STATISTICS
C   R1(K): COEFFICIENTS of ESTIMATION
C   RX(K):  THE VARIANCES of ORDER
   STATISTICS USED
C   NUM:  NUMERATOR, EXP:  DE:
   DENOMINATOR
C   II(K):  RANKS of ORDER STATISTICS
   USED
C   SC(K), c: SPACING, S(K): RANK
C   A:  BLOM'S CONSTANT
C   K1,K2:  THE STARTING, FINISH and
   NUMBER of SPACING for N
C   IW:  SWITCH
C   VAR:  VARIANCE of ESTIMATION
C   AEF: ASYMPTOTIC EFFICIENCY
C   KK:  SWITCH of INTERACTIVE
   INPUT
C   KK2:  SWITCH of SPACING and INPUT
   for ALL SIZE from N0 to N6
C   COP:  COEFFICIENTS of BLUE
C   Vopt: VARIANCE of BLUE
C   R.E.: EFFICIENCY Vopt/VAR
C   WRITE  FOUR DISK FILES
C   FUNCTION:  <1> DISPLAY and SAVE
   THE VECTOR of EXPECTATION and THE
   MATRIX of
C   COVARIANCES of ORDER STATISTICS
   USED
C   <2> COMPUTE COEFFICIENTS of
   ESTIMATION ACCORDING TO THE
   PROPORTION
C   of ITS COEFFICIENTS
C   <3> COMPUTE THE VARIANCE and
   EFFICIENCY of ESTIMATION
C   <4> RANK of ORDER STATISTICS MAY
   BE OBTAINED by THE SPACING and
C   THE INTERACTIVE INPUT

CCC   (1): SPACING:
   SC(1)=.10293
   SC(2)=.25
   SC(3)=.125
   SC(4)=0.0625
   m=5-2=3, K1=2, M1=4, K2=5
   SC(5)=0.03125
   SC(6)=1./FLOAT(2**6)
   SC(7)=1./FLOAT(2**7)
   SC(8)=1./FLOAT(2**8)
```

SC(9)=1./FLOAT(2**9)
SC(10)=1./FLOAT(2**10)

C WRITE(*,*)' SPACING: ',SC
CCCC (2): COMPUTE THE N1,N2,N3,N4,N5
and RN1,RN2,RN3,RN4,RN5

a=.5
I=1
DO 10 L=2,10
    c=SC(L)
    DO 20 N=2,1024
        IF((c*(FLOAT(N)+1.-2.*a)+a) .GE. FLOAT(I)) GO TO 25
    20   CONTINUE
    25   RN1(L)=N
        N1(L)=N
        DO 30 N=2,1024
            IF((c*(FLOAT(N)+1.-2.*a)+a).GE. FLOAT(I+1)) GO TO 35
    30   CONTINUE
C    PAUSE'WARNING !'
    35   RN2(L)=N-1
        N2(L)=N-1
        N3(L)=(N1(L)+N2(L)+1)/2
        RN3(L)=N3(L)
        RN4(L)=N4(L)
        N5(L)=(N1(L)+N3(L))/2
        RN5(L)=N5(L)
10   CONTINUE
CCCC (3): THE START, FINISH and
NUMBER of SPACING: K1,K2 for N

OPEN(1,FILE='LOGIS1.EXP',STATUS='OLD')
OPEN(2,FILE='LOGIS1.COV',STATUS='OLD')
OPEN(4,FILE='LGEST1.VAR',STATUS='NEW')

KK2=0
WRITE(*,*)' KK2=0    : FINDING RANK
by SPACING'
WRITE(*,*)' KK2=1 : FINDING RANK
by INTERACTIVE INPUT'
WRITE(*,*)' KK2=2: OTHER FINDING RANK
by VALUE KK, ONE by ONE'
WRITE(*,*)' THERE ARE THE BLUES
WHEN AND ONLY WHEN KK2=2 OR KK=2'
WRITE(*,*)' INPUT KK2=?'
READ(*,*) KK2
WRITE(*,*)' INPUT THE STARTING and
FINISH N0,N6 =?'
READ(*,*)N0,N6
48 FORMAT(2X,' THE STARTING and
FINISH of SIZE: N0,N6=',2I4)
WRITE(*,48)N0,N6
DO 170 N=N0,N6
    RN=FLOAT(N)
    K1=2
    DO 50 L=2,10
        RN=FLOAT(N)
        K1=2
        DO 50 L=2,10
            C THE START, FINISH and NUMBER of
            SPACING: K1,K2 for N
            IF(N,GE,N5(K2)) K2=2
    50   CONTINUE
70 FORMAT(' N,N5(K2),K1,K2=',5I4)
WRITE(*,70)N,N5(K2),K1,K2
DO 65 L=K1,K2
    60 FORMAT(' N,N5(K2),K1,K2=',5I4)
WRITE(*,60)N,N5(K2),K1,K2
65 CONTINUE
CC PDF  f(x)=exp(-x)/[(1 +exp(-x))**2]
C CDF  F(x)=1/(1 +exp(-x))
C with \theta=TAO=0.55132890  G(u) =X=ln[(1-
        u)/u]=ln[u/(1-u)]
C                      G'(u) =1/[u(1-u)]
C                 G"(u) =-(1-2u)/[((u(1-u))**2]
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C             1/2**L
C
C      N1  N5   N3    N2  N4
C      RN1 RN5  RN3   RN2 RN4
C
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
CCCC (4):

KK=1
    IF(KK2.EQ.1) GO TO 90
    IF(KK2.EQ.2) GO TO 77
WRITE(*,*)' INPUT CF1(K)? NO/YES 1/2'
READ(*,*)IW
    IF(IW.EQ.1) GO TO 90
77   KK=2
    WRITE(*,*)' INPUT NUMBER of ORDER
STATISTICS USED, M2?'
READ(*,*)M2
    M1=(M2+1)/2
    DO 75 K=1,M1
        W(K)=1.
        78   FORMAT('IJ(',I2,')=?',' CF1(',I2,')=?')
            WRITE(*,78)K,K
            READ(*,*)IJ(K),CF1(K)
    75   CONTINUE
82   CONTINUE
WRITE(*,86)CF
    86   FORMAT(2X,' CF:',5F9.4/,'      ',5F9.4)
    84   FORMAT('CF(',I2,')=?')
        WRITE(*,84)K
        READ(*,*)CF(K)
82   CONTINUE
WRITE(*,86)CF
    86   FORMAT(2X,' CF:',5F9.4/,'      ',5F9.4)
CCCC (5): THE RANK S(L),CF1(K),W(K),
IJ(K)
RN=N
DO 100 L=K2,K1,-1
S(L)=INT(RN*SC(L)+0.5)
98 FORMAT(2X,'S(',I3,')=',I3)
WRITE(*,98)L,S(L)
100 CONTINUE
CF1(1)=CF(K2)
CF1(2)=CF(K2)
W(1)=1.
W(2)=1.
IJ(1)=S(K2)
IJ(2)=S(K2)+1
K=2
DO 110 L=K2-1,K1,-1
IF(S(L+1).EQ.S(L)) THEN
CF1(K-1)=CF1(K-1)+CF(L)
CF1(K)=CF1(K)+CF(L)
W(K-1)=W(K-1)+1.
W(K)=W(K)+1.
GO TO 110
ELSE
ENDIF
IF((S(L+1)+1).EQ.S(L)) THEN
CF1(K)=CF1(K)+CF(L)
W(K)=W(K)+1.
K=K+1
IJ(K)=S(L)+1
CF1(K)=CF(L)
W(K)=1.
GO TO 110
ELSE
ENDIF
110 CONTINUE
M1=K
M2=2*K1
120 DO 130 J=1,M1
W(M2+1-J)=W(J)
IF((M2+1-J).EQ.J) CF1(J)=.0
CF1(M2+1-J)=-CF1(J)
IJ(M2+1-J)=N+1-IJ(J)
130 CONTINUE
140 FORMAT(10I4)
150 FORMAT(1X,' N=',I3)
160 FORMAT(10F9.5)
WRITE(*,150)N
WRITE(*,'(*)') IJ,CF1,W=''
WRITE(*,140)(IJ(K),K=1,M2)
WRITE(*,160)(CF1(K),K=1,M2)
WRITE(*,160)(W(K),K=1,M2)
CCCC (6): THE EXPECTATION of ORDER
STATISTICS USED
180 FORMAT(2I4,F9.5)
DO 200 I=1,M1
195 READ(1,180)M,I2,BB1
IF((M.EQ.N).AND.(I2.EQ.IJ(I))) GO TO 210
GO TO 195
210 R(I)=BB1
200 CONTINUE
DO 220 K=1,M1
R(M2+1-K)=-R(K)
220 CONTINUE
CALL VLTV(R,M1,'EXPECT-L',' 20F9.4 ')
CCCC (7): FIND OUT THE COVARIANCES
of ORDER STATISTICS USED
220 FORMAT(3I4,F9.5)
DO 224 I=1,M1
DO 228 J=I,M2+1-I
230 READ(2,220)M,I2,J2,BB2
IF((M.EQ.N).AND.(I2.EQ.IJ(I)).AND.(J2.EQ.IJ
(J))) GO TO 240
GO TO 230
C SPACING: from K1=2 to K2
240 RX((J-1)*J/2+I)=BB2
RX((M2+1-I-1)*(M2+1-I)/2+M2+1-J)=BB2
228 CONTINUE
224 CONTINUE
CCCC LAST TWO LINES of COV-MATRIX
DO 234 K=1,M2
RX((M2*(M2+1)/2 +K)=R(K)
IF(K.GT.M1) RX(M2*(M2+1)/2 +K)=-
R(M2+1-K)
234 CONTINUE
RX((M2+1)*(M2+2)/2)=RN
DO 238 K=1,M2
RX((M2+1)*(M2+2)/2 +K)=IJ(K)
238 CONTINUE
RX((M2+1)*(M2+2)/2 +M2+1)=K1
RX((M2+2)*(M2+3)/2)=K2
CALL VARML(RX,M2+2,'COV---RX',
8F9.5 ',1)
CCCC (8): OUTPUT
WRITE(*,'(*)') DISPLAY THE LOWER-
TRIANGULAR MATRIX of COV YES/NO 1/2'
READ(*,*)IW
IF(IW.NE.1) GO TO 245
CALL VARML(RX,M2+2,'COV---RX',
8F9.5 ',1)
245 CONTINUE
WRITE(*,*)'WRITE COV FILE? YES/NO'
1/2
READ(*,*)IW
IF(IW.NE.1) GO TO 255
WRITE(*,*)'INPUT THE FILE NAME (LG??.COV) '
READ(*,260)FNAME
260  FORMAT(A)
OPEN(3,FILE=FNAME,STATUS='NEW')
270  FORMAT(I4)
CALL VARML(RX,M2+2,'COV---RX',8F9.5',2)
CLOSE(3)
255  NUM=.0
DO 280 I=1,M1
  DO 295 K=1,I
    NUM=NUM+W(I)*CF1(K)*RX((I-1)*I/2+K)*CF1(I)*W(I)
  295  CONTINUE
  DO 300 K=I+1,M2
    NUM=NUM+W(K)*CF1(K)*RX((K-1)*K/2+I)*CF1(I)*W(I)
  300  CONTINUE
280  CONTINUE
  EXP=0
  DO 310 K=1,M1
   EXP=EXP+W(I)*CF1(K)*R(K)
  310  CONTINUE
  EXP=-2.*EXP
   DO 320 K=1,M1
     R1(K)=CF1(K)/EXP
     R1(M2+1-K)=R1(K)
  320  CONTINUE
   CALL VLTV(R1,M2,'COEFIC-L',20F9.4')
   DE=EXP*EXP
   NUM=2.*NUM
   VAR=NUM/DE
   WRITE(*,*)'By Using The Cramer-Rao Lower Bound, We Obtain:'
   AEF=(9./(3.+3.1415926**2))/(RN*VAR)
330  FORMAT(2X,'N=',I3,'EXP=',F9.5,'NUM=',F9.5,'VAR=',F7.5,
     / ' AEF=',F7.5,
     WRITE(*,,330)N,EXP,NUM,VAR,AEF
WRITE(4,330)(R1(K),K=1,M1)
340  FORMAT(2X,'IJ(K):',10(6X,I3)/,')
WRITE(4,340)(R1(K),K=1,M1)
350  FORMAT('IJ(K):',10(6X,I3)/,'R1(K): ',10F9.5/)
WRITE(4,350)(R1(K),K=1,M1)
360  FORMAT('Vopt=','F9.4,' R.E.=Vopt/VAR=','F7.5)
WRITE(*,360)Vopt,Vopt/VAR
WRITE(4,360)Vopt,Vopt/VAR
CALL VLTV(COP,M2,'COEFIC-L',20F9.4')
370  FORMAT(' C(OP(K))','F9.4,',' F9.4)
WRITE(4,370)(COP(K),K=1,M2)
CALL VARML(RY,M2,'INV-MATR','8F9.5',1)
375  CONTINUE
CC TURN TO NEXT SIZE N
170  CONTINUE
   CLOSE(1)
   CLOSE(2)
   CLOSE(4)
   STOP
END
C Cholesky's Inverse Matrix and Lloyd's Optimal-Method (as Ogawa Method)
SUBROUTINE MINV(RX,RZ,R,COP,V)
DIMENSION RX(*),RY(820),R(M),COP(M),S(100,100)
REAL SUM,Vopt,RZ(*)
INTEGER M,K,I,J,L
S(1,1)=SQRT(RX(1))
DO 10 J=2,M
   S(1,J)=RX((J-1)*J/2+1)/S(1,1)
10   CONTINUE
DO 20 I=2,M
   L=I-1
   SUM=0
   DO 30 K=1,L
      SUM=SUM+S(K,I)*S(K,I)
   30   S(I,I)=SQRT(RX((I-1)*I/2+I)-SUM)
   DO 20 J=1,M
      SUM=0
      IF(I-J) 25,20,35
25   S(I,J)=0
20   CONTINUE
WRITE(*,*)'M=',M
RY(1)=1./S(1,1)
DO 50 J=2,M
   RY((J-1)*J/2+1)=1./S(J,J)
   SUM=0
   L=J-1
   DO 60 I=1,L
      SUM=SUM+RY((I-1)*I/2+1)*S(I,J)
   60   RY((J-1)*J/2+1)=-SUM/S(J,J)
   DO 70 I=2,M
      DO 70 J=1,M
         IF(I-J) 65,70,75
            SUM=0
            L=J-1
35   S(L,J)=0
20   CONTINUE
WRITE(*,*)'By Using The Cramer-Rao Lower Bound, We Obtain:'
   AEF=(9./(3.+3.1415926**2))(RN*VAR)
330  FORMAT(2X,'N=',I3,'EXP=',F9.5,'NUM=',F9.5,'VAR=',F7.5,
     / ' AEF=',F7.5,
     WRITE(*,,330)N,EXP,NUM,VAR,AEF
WRITE(4,330)(R1(K),K=1,M1)
340  FORMAT(2X,'IJ(K):',10(6X,I3)/,')
WRITE(4,340)(R1(K),K=1,M1)
350  FORMAT('IJ(K):',10(6X,I3)/,'R1(K): ',10F9.5/)
WRITE(4,350)(R1(K),K=1,M1)
360  FORMAT('Vopt=','F9.4,' R.E.=Vopt/VAR=','F7.5)
WRITE(*,360)Vopt,Vopt/VAR
WRITE(4,360)Vopt,Vopt/VAR
CALL VLTV(COP,M2,'COEFIC-L',20F9.4')
370  FORMAT(' C(OP(K))','F9.4,',' F9.4)
WRITE(4,370)(COP(K),K=1,M2)
CALL VARML(RY,M2,'INV-MATR','8F9.5',1)
375  CONTINUE
CC TURN TO NEXT SIZE N
170  CONTINUE
   CLOSE(1)
   CLOSE(2)
   CLOSE(4)
   STOP
END
C Cholesky's Inverse Matrix and Lloyd's Optimal-Method (as Ogawa Method)
SUBROUTINE MINV(RX,RZ,R,COP,V)
DIMENSION RX(*),RY(820),R(M),COP(M),S(100,100)
REAL SUM,Vopt,RZ(*)
INTEGER M,K,I,J,L
S(1,1)=SQRT(RX(1))
DO 10 J=2,M
   S(1,J)=RX((J-1)*J/2+1)/S(1,1)
10   CONTINUE
DO 20 I=2,M
   L=I-1
   SUM=0
   DO 30 K=1,L
      SUM=SUM+S(K,I)*S(K,I)
   30   S(I,I)=SQRT(RX((I-1)*I/2+I)-SUM)
   DO 20 J=1,M
      SUM=0
      IF(I-J) 25,20,35
25   S(I,J)=0
20   CONTINUE
WRITE(*,*)'M=',M
RY(1)=1./S(1,1)
DO 50 J=2,M
   RY((J-1)*J/2+1)=1./S(J,J)
   SUM=0
   L=J-1
   DO 60 I=1,L
      SUM=SUM+RY((I-1)*I/2+1)*S(I,J)
   60   RY((J-1)*J/2+1)=-SUM/S(J,J)
   DO 70 I=2,M
      DO 70 J=1,M
         IF(I-J) 65,70,75
            SUM=0
            L=J-1
35   S(L,J)=0
20   CONTINUE
WRITE(*,*)'By Using The Cramer-Rao Lower Bound, We Obtain:'
   AEF=(9./(3.+3.1415926**2))(RN*VAR)
330  FORMAT(2X,'N=',I3,'EXP=',F9.5,'NUM=',F9.5,'VAR=',F7.5,
     / ' AEF=',F7.5,
     WRITE(*,,330)N,EXP,NUM,VAR,AEF
WRITE(4,330)(R1(K),K=1,M1)
340  FORMAT(2X,'IJ(K):',10(6X,I3)/,')
WRITE(4,340)(R1(K),K=1,M1)
350  FORMAT('IJ(K):',10(6X,I3)/,'R1(K): ',10F9.5/)
WRITE(4,350)(R1(K),K=1,M1)
`90   CONTINUE
90   CONTINUE
90   CONTINUE
CC(4) LOWER TRIANGLE RY of S
   CALL VARML(RZ,M,'MAT--INV', 8F9.4 ','1)
   CALL VARML(RY,M,'MAT--INV', 8F9.4 ','1)
CC(3) INVERSE MATRIX S of RX
   DO 90 I=1,M
   DO 100 J=1,M
   DO 92 K=1,M
   COP(K)=.0
   IF(K.LT.J) GO TO 92
   COP(K)=RY((K-1)*K/2 +J)
   92   CONTINUE
   SUM=.0
   DO 96 K=1,M
   IF(K.LT.I) GO TO 96
   SUM=SUM +COP(K)*RY((K-1)*K/2 +I)
   96   CONTINUE
   S(I,J)=SUM
   IF(I.GT.J) GO TO 100
   RZ((J-1)*J/2 +I)=S(I,J)
   100  CONTINUE
   CALL VARML(RZ,M,'MAT--INV', 8F9.4 ','1)
CC(5) CHECK:
   DO 120 I=1,M
   DO 130 J=1,M
   SUM=.0
   DO 140 K=1,J
   SUM=SUM+RX((J-1)*J/2 +K)*S(I,K)
   140  CONTINUE
   DO 145 K=J+1,M
   SUM=SUM+RX((K-1)*K/2 +J)*S(I,K)
   145  CONTINUE
   IF(I.GT.J) GO TO 148
   RY((I-1)*I/2 +J)=SUM
   148  CONTINUE
   130  CONTINUE
   120  CONTINUE
CC(6) Lloyd Method:
   DO 150 I=1,M
   COP(I)=.0
   DO 160 K=1,M
   COP(I)=COP(I) +R(K)*S(I,K)
   160  CONTINUE
   Vopt=1./Vopt
   CALL VLTV(COP,M,'COEF-OPT', 6F9.4')
   RETURN
END
C
SUBROUTINE VARML(A,M,TITLE,FM,IW)
   INTEGER M,IW
   REAL A(*)
   CHARACTER*8 TITLE,FM
   CHARACTER F*48,FF(6)*8
   EQUIVALENCE (F,TITLE,FM)
   DATA F/' (1X,I5,2H) ,   /(8X,           )) '/
   IF(IW.EQ.1) WRITE(*,F) I,(A(IJ),IJ=I1,II)
   IF(IW.EQ.2) WRITE(3,F) I,(A(IJ),IJ=I1,II)
   10 CONTINUE
   800 FORMAT(/1X,2A4,4X,1H(,I5,3H*
   805 FORMAT(/1X,A8,4X,1H(,I5,2H ))/(1X,
C
SUBROUTINE VLTV(V,N,TITLE,FM)
   INTEGER N
   REAL V(N)
   CHARACTER*8 TITLE,FM
   DATA F/'  (/1X,A8,4X,1H(,I5,2H )/(1X,
   WRITE(*,F) TITLE,N,(V(I),I=1,N)
   RETURN
END
C
SUBROUTINE VARML(A,M,TITLE,FM,IW)
   INTEGER M,IW
   REAL A(*)
   CHARACTER*8 TITLE,FM
   DATA F/' /1X,A8,4X,1H(,I5,3H
   IF(IW.EQ.1) WRITE(*,805) TITLE,M,M
   IF(IW.EQ.2) WRITE(3,805) TITLE,M,M
   II = 0
   DO 10 I = 1,M
   I1 = II+1
   II = II+I
   IF(IW.EQ.1) WRITE(*,F) I,(A(IJ),IJ=I1,II)
   IF(IW.EQ.2) WRITE(3,F) I,(A(IJ),IJ=I1,II)
   10 CONTINUE
   800 FORMAT(/1X,2A4,4X,1H(,I5,3H*
   805 FORMAT(/1X,A8,4X,1H(,I5,2H ))/(1X,
   RETURN
END
C
SUBROUTINE VLTV(V,N,TITLE,FM)
   INTEGER N
   REAL V(N)
   DATA F/' /1X,A8,4X,1H(,I5,3H
   WRITE(*,F) TITLE,N,(V(I),I=1,N)
   RETURN
END
C
SUBROUTINE VARML(A,M,TITLE,FM,IW)
   INTEGER M,IW
   REAL A(*)
   CHARACTER*8 TITLE,FM
   DATA F/' /1X,A8,4X,1H(,I5,3H
   IF(IW.EQ.1) WRITE(*,805) TITLE,M,M
   IF(IW.EQ.2) WRITE(3,805) TITLE,M,M
   II = 0
   DO 10 I = 1,M
   I1 = II+1
   II = II+I
   IF(IW.EQ.1) WRITE(*,F) I,(A(IJ),IJ=I1,II)
   IF(IW.EQ.2) WRITE(3,F) I,(A(IJ),IJ=I1,II)
   10 CONTINUE
   800 FORMAT(/1X,2A4,4X,1H(,I5,3H*
   805 FORMAT(/1X,A8,4X,1H(,I5,2H ))/(1X,
   RETURN
END
C
SUBROUTINE VLTV(V,N,TITLE,FM)
   INTEGER N
   REAL V(N)
   DATA F/' /1X,A8,4X,1H(,I5,3H
   WRITE(*,F) TITLE,N,(V(I),I=1,N)
   RETURN
END