A Remark on the Convolution of the Generalised Logistic Random Variables

Matthew Oladejo Ojo *
Department of Mathematics,
Obafemi Awolowo University,
Ile-Ife, Nigeria.

Abstract

In this paper, the distribution of the sum of independent random variables from the generalized logistic distribution is determined by using a much simpler and shorter method than had earlier been used.

1. INTRODUCTION

A generalization of the logistic random variable whose probability density function and the characteristic function are given respectively as

\[ f(x) = \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)} \frac{e^{px}}{(1+e^x)(p+q)}, \quad -\infty < x < \infty, \quad p > 0, \quad q > 0 \]

(1.1)

*Research supported by the University Research Committee of Obafemi Awolowo University, Ile-Ife, Nigeria.
and
\[ \Psi(s) = \frac{\Gamma(p+is)\Gamma(q-is)}{\Gamma(p)\Gamma(q)} \] (1.2)
has earlier been considered by George and Ojo (1980). They evaluated the cumulants and showed how its distribution function can be approximated by the t-distribution.

The importance of the sampling distribution of the sample mean of a population is well known. On the basis of this, the distribution of the sum of independent and identically distributed random variables from a generalised logistic population has been determined in an earlier paper Ojo and Adeyemi, (1989) by using the Mittag-Leffler expansion of the characteristic function. In this paper a much simpler and shorter method is used to obtain the same distribution.

2. THE PROBABILITY DENSITY FUNCTION OF THE CONVOLUTION OF THE GENERALISED RANDOM VARIABLES

Let \( X_1, \ldots, X_n \) be \( n \) independent and identically distributed random variables each having the distribution (1.1). Let
\[ Y = \sum_{i=1}^{n} X_i \]
We want to determine the distribution of \( Y \). The characteristic function of \( Y \) is given as
\[ \Psi_n(s) = \frac{(\Gamma(p+is))^n(\Gamma(q-is))^n}{(\Gamma(ip))^n(\Gamma(q))^n} \]
and by inversion formula, the density function of \( Y \) is given as
\[ f_n(y) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-isy}\Psi_n(s)ds \]
\[ = (\Gamma(p))^{-n}(\Gamma(q))^{-n} \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-isy}(\Gamma(p+is))^n(\Gamma(q-is))^n ds \] (2.1)
Now if we let \( p + is = -z \), we have
\[ f_n(y) = e^{py}(\Gamma(p))^{-n}(\Gamma(q))^{-n} \frac{1}{2\pi i} \int_{-\infty}^{-\infty} e^{piz}(\Gamma(-z))^n(\Gamma(z+p+q))^n dz \] (2.2)
If \( g(z) \) denotes the integrand in (2.2). It can be shown that \( |z^m g(z)| \to 0 \) as \( z \to \infty \) so that the integral converges and then

\[
f_n(y) = e^{py}(\Gamma(p))^{-n}(\Gamma(q))^{-n}
\]

\[
\times \frac{1}{2\pi i} \int_C e^{py}(\Gamma(-z))^{-n}(\Gamma(z + p + q))^{-n}dz
\]

(2.3)

where \( C \) is the contour bounded by the line \( x = -p \) and that part of the circle \( |z| = m + \frac{1}{2}, \ m \to \infty \) which lies to the right of the straight line. The contour is then transversed in a counter-clockwise direction.

Now

\[
\Gamma(-z) = \frac{-\pi}{(\Gamma(z + 1)\sin \pi z)}
\]

So that (2.3) becomes

\[
f_n(y) = -e^{py} \pi^n(\Gamma(p))^{-n}(\Gamma(q))^{-n}(-1)^n
\]

\[
\times \frac{1}{2\pi i} \int_C (\Gamma(z + 1))^{-n}(\Gamma(z + p + q))^{-n}(\sin \pi z)^{-n}e^{yz}dz
\]

The poles of the integrand in (2.4) are of \( n^{th} \) order and are those of \( \Gamma(-z) \), viz, \( z = r, \ r = 0, 1, 2, \ldots \) and so the integrand in (2.4) is \( 2\pi i \) times the sum of the residues of the poles within the contour \( C \). Thus the density function of \( Y \) is given as

\[
f_n(y) = e^{py}(\Gamma(p))^{-n}(\Gamma(q))^{-n}
\]

\[
\times \sum_{r=0}^{\infty} \frac{(-1)^{n+r+1}}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} \left[ e^{yz}(\Gamma(z + p + q))^n (\Gamma(z + 1))^{-n} \right]_{z=r}
\]

This density function can be evaluated for fixed values of \( p \) and \( q \). For example when \( p = q = 1 \), we get

\[
f_n(y) = \sum_{r=0}^{\infty} \frac{(-1)^{n+r+1}}{(n-1)!} e^y \left[ \frac{d^{n-1}}{dz^{n-1}} e^{yz}(z + 1)^n \right]_{z=r}
\]

When \( n = 1 \), the density function (2.4) reduces to
which is to be expected. For higher values of $n$ and various combinations of $p$ and $q$, the infinite series form obtained for the density function can be expressed as a finite term by using the following lemma.

Lemma 2.1

Let $m$ be a positive integer. Then

\[(a)\quad \sum_{k=1}^{\infty} k^m e^{ky} = \frac{d^m}{dy^m} \left( \frac{e^y}{1 - e^y} \right)\]

\[(b)\quad \sum_{k=1}^{\infty} (-1)^{k-1} k^m e^{ky} = \frac{d^m}{dy^m} \left( \frac{e^y}{1 + e^y} \right)\]

\[(c)\quad \sum_{k=1}^{\infty} k^m e^{-ky} = (-1)^m \frac{d^m}{dy^m} \left( \frac{e^{-y}}{1 - e^{-y}} \right)\]

\[(d)\quad \sum_{k=1}^{\infty} (-1)^{k-1} k^m e^{-ky} = (-1)^m \frac{d^m}{dy^m} \left( \frac{e^{-y}}{1 + e^{-y}} \right)\]

Proof: The proof of the lemma follows immediately by differentiating $m$ times the following identities

\[(i)\quad \sum_{k=1}^{\infty} e^{ky} = \frac{e^y}{1 - e^y}\]

\[(ii)\quad \sum_{k=1}^{\infty} (-1)^{k-1} e^{ky} = \frac{e^y}{1 + e^y}\]

\[(iii)\quad \sum_{k=1}^{\infty} e^{-ky} = \frac{e^{-y}}{1 - e^{-y}}\]

\[(iv)\quad \sum_{k=1}^{\infty} (-1)^{k-1} e^{-ky} = \frac{e^{-y}}{1 + e^{-y}}\]

The distribution function of $Y$ can be evaluated for various combinations of $p$ and $q$. For instance when $p = q = 1$, the distribution function is given by

\[F_n(y) = \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} \frac{(-1)^{n+1}}{(n-1)!} \binom{n-1}{k} \frac{j!}{(j-n+k+1)!}\]
\[
\sum_{r=0}^{\infty} (-1)^{nr+j+k+1-n} e^{(r+1)y} \sum_{s=0}^{k} \frac{(-1)^s k! y^{k-s}}{(k-s)!}
\]

When \( n = 1 \), the distribution function reduces to

\[
F_n(y) = \frac{e^y}{1 + e^y}
\]

which is to be expected.

References
