On Characterizations of the Half Logistic Distribution

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Abstract

Characterizations of the probability distributions have been discussed in the literature and many probability distributions have been well characterized. Among the probability distributions which are yet to be fully characterized is the half logistic distribution. In this paper, some theorems that characterize the half logistic distribution are stated and proved. A possible application of one of the characterization theorems is included.

KEY WORDS: Half Logistic distribution, Characterizations, Exponential distribution, Pareto distribution, Uniform distribution, Homogeneous Differential Equation.

1 Introduction

A random variable Y is said to have an half logistic distribution if its probability density function is given as

$$f_Y(y) = \frac{2e^y}{(1+e^y)^2}, \qquad 0 < y < \infty \tag{1.0.0}$$

and its probability distribution function is

$$F_Y(y) = \frac{e^y - 1}{(1 + e^y)}, \qquad 0 < y < \infty. \tag{1.0.1}$$

Balakrishnan [1] has established some recurrence relations for the moments and product moments of order statistics for half logistic distribution given in the equation (1.0.0).

If a location parameter μ and a scale parameter σ are introduced in the equation (1.0.0), the density function of the random variable Y becomes

$$f_Y(y; \mu, \sigma) = \frac{2exp(\frac{y-\mu}{\sigma})}{\sigma\{1 + exp(\frac{y-\mu}{\sigma})\}^2}, \qquad \mu \leq y < \infty, \quad \sigma > 0. \tag{1.0.2}$$

Balakrishnan and Puthenpura [2] have obtained the best linear unbiased estimates of the parameters μ and σ. They also tabulated the values of the variance and covariance of these estimators.

The characterizations of this distribution have not received much attention since it was derived from the logistic distribution. Though the characterizations of logistic and log-logistic distributions have received some appreciable attention. Ojo [6], Olapade and Ojo [7] gave some theorems on characterizations of logistic distribution and generalized logistic distribution while Mohammed, Karan and Alfred [5] worked on characterization of generalized log-logistic distribution.

Analogous to the characterizations of the logistic and log-logistic distributions, one may find some uses of the characterizations of the half logistic distribution. These problems need to be investigated further and are currently being looked into.

In this paper, we state some theorems that characterize the half logistic distribution and we prove them.

1.1 Some needed distribution functions:

For the purpose of this paper, we recall the probability density functions of some known distributions that will appear in the theorems of the next section.

(i) The Pareto distribution: McDonald and Xu [4] show a two parameters Pareto distribution with the probability density function

$$f_X(x; b, p) = \frac{pb^p}{x^{p+1}}, \quad x > b \tag{1.1.0}$$

where p is a positive integer. The ordinary Pareto distribution density function with $b = 2, p = 1$ is

$$f_X(x) = \frac{2}{x^2}, \quad x > 2. \tag{1.1.1}$$

(ii) The exponential distribution: A well known exponential distribution with parameter λ has the probability density function

$$f_X(x; \lambda) = \lambda e^{-\lambda x}, \quad x > 0, \quad \lambda > 0. \tag{1.1.2}$$

(iii) The Uniform distribution: The probability density of a random variable X that has uniform distribution in its simplest form is given as

$$f_X(x) = 1, \quad 0 \leq x \leq 1. \tag{1.1.3}$$

2 Some Characterizations:

We state some characterization theorems and prove them in this section.

Theorem 1: Let X be a continuously distributed random variable with probability density function $f_X(x)$. The random variable $Y = ln(\frac{1+e^{-x}}{1-e^{-x}})$ is an half logistic random variable if and only if X follows an exponential distribution with parameter $\lambda = 1$.

Proof: If X is exponentially distributed with parameter 1,

$$f_X(x) = e^{-x}, \quad 0 < x < \infty. \tag{2.1.0}$$

Let $y = ln(\frac{1+e^{-x}}{1-e^{-x}})$, by transformation of random variable $x = ln(\frac{1+e^{y}}{e^{y}-1})$.

But

$$f_Y(y) = |\frac{d}{dy}(g^{-1}(y))| f_X(y) = |\frac{d}{dy}(x)| f_X(y).$$

So, we arrive at

$$f_Y(y) = \frac{2e^y}{(1+e^y)^2} \tag{2.1.1}$$

which is the probability density function of the half logistic distribution.

Conversely, suppose the random variable Y is an half logistic random variable with the distribution function shown in the equation (1.0.1), then

$$F_X(x) = Pr[X \leq x] = Pr[ln(\frac{e^y + 1}{e^y - 1}) \leq x]$$

$$= 1 - Pr[y \leq ln(\frac{e^x + 1}{e^x - 1})] = 1 - e^{-x} \tag{2.1.2}$$

which is the distribution function of the exponential distribution with parameter 1. This completes the proof.

Theorem 2: Let a continuously distributed random variable X has uniform distribution over the unit range (0,1), then the random variable $Y = ln(\frac{a-X}{X})$ has an half logistic distribution if and only if $a = 2$.

Proof: Suppose X is uniformly distributed

$$f_X(x) = 1, \quad 0 < x < 1. \tag{2.2.0}$$

Let $y = ln(\frac{a-x}{x})$, then by the transformation of random variable, $x = \frac{a}{1+e^y}$ and the Jacobian of the transformation is

$$|J| = \frac{ae^y}{(1 + e^y)^2}. \tag{2.2.1}$$

Thus the density function of Y is

$$f_Y(y) = \frac{2e^y}{(1 + e^y)^2} \tag{2.2.2}$$

when $a = 2$, which is the density function of the half logistic random variable Y.

Conversely, suppose Y is an half logistic random variable with the probability distribution function in the equation (1.0.1), then

$$F_X(x) = Pr[X \leq x] = Pr[\frac{a}{1 + e^y} \leq x]$$

$$= 1 - Pr[y \leq ln(\frac{a - x}{x})] = x \text{ for } a = 2. \tag{2.2.3}$$

Since this is the probability distribution function of the uniform random variable, the theorem is proved.

Theorem 3: Let X be a continuously distributed random variable with probability density function $f_X(x)$. Then the random variable $Y = ln(x-1)$ is an half logistic random variable if and only if X follows a Pareto distribution with parameters $p = 1$ and $b = 2$.

Proof: If X has the Pareto distribution with parameters $p = 1$ and $b = 2$, then

$$f_X(x) = \frac{2}{x^2}, \quad x > 2. \tag{2.3.0}$$

The random variable $y = ln(x - 1)$ implies that $x = 1 + e^y$ and the Jacobian of the transformation is $|J| = e^y$. Therefore

$$f_Y(y) = \frac{2e^y}{(1 + e^y)^2} \tag{2.3.1}$$

which is the probability density function of the half logistic distribution.

Conversely, if Y has an half logistic distribution with probability distribution function shown in equation (1.0.1), then

$$\begin{aligned} F_X(x) = Pr[X \leq x] = Pr[1 + e^y \leq x] = F_Y[ln(x - 1)] \\ = 1 - \frac{2}{x}. \end{aligned} \tag{2.3.2}$$

Since the equation (2.3.2) is the probability distribution function for the Pareto distribution given in the equation (2.3.0), the theorem is proved.

Theorem 4: Let X be a continuously distributed random variable with probability density function $f_X(x)$. Then the random variable $Y = ln(2e^x - 1)$ is an half logistic random variable if and only if X follows an exponential distribution with parameter $\lambda = 1$.

Proof: Suppose X has exponential distribution with parameter $\lambda = 1$,

$$f_X(x) = e^{-x}, \quad x > 0. \tag{2.4.0}$$

Let $y = ln(2e^x - 1)$, this implies that $x = ln(\frac{1+e^y}{2})$. So, the Jacobian of the transformation is

$$|J| = \frac{e^y}{1 + e^y}. \tag{2.4.1}$$

Therefore,

$$f_Y(y) = \frac{2e^y}{(1+e^y)^2} \tag{2.4.2}$$

which is the probability density function of the half logistic distribution.

Conversely, if Y is an half logistic random variable, then the distribution function of Y is as given in the equation (1.0.1). But

$$F_X(x) = Pr[X \leq x] = Pr[y \leq ln(2e^x - 1)]$$

$$= 1 - e^{-x}. \tag{2.4.3}$$

Since the equation (2.4.3) is the distribution function of the exponential distribution when $\lambda = 1$, the theorem is proved.

Theorem 5: The random variable Y is half logistic with probability density function f given in the equation (1.0.0) if and only if f satisfies the homogeneous differential equation

$$2e^y f' + (e^{2y} - 1)f^2 = 0 \tag{2.5.0}$$

(prime denotes differentiation).

Proof:

Since

$$f_Y(y) = \frac{2e^y}{(1+e^y)^2}, \quad F(y) = \frac{e^y - 1}{(1+e^y)}, \quad f' = \frac{2e^y(1-e^y)}{(1+e^y)^3}.$$

Suppose Y is half logistic, it is easily shown that $f_Y(y)$ above satisfies the equation (2.5.0).

Conversely, if we assume that f satisfies (2.5.0). Separating the variables in the equation (2.5.0) and integrating, we have $f = 2(e^y + e^{-y} - k)^{-1}$ where k is a constant. The value of k that makes f a density function is $k = -2$.

Possible Application of Theorem 5: From the equation (2.5.0), we have

$$y = ln(\frac{-f' + \surd(f'^2 + f^4)}{f^2})$$

or equivalently

$$y = ln(\frac{-F'' + \surd(F''^2 + F'^4)}{F'^2}) \tag{2.5.1}$$

where F is the corresponding distribution function as shown above. Thus, the importance of the theorem (5) lies in the linearising transformation (2.5.1). The transformation (2.5.1) can be regarded as an half logistic model alternative to Berkson's logit transform (Berkson [3]) and Ojo's logit transform when $p = q = 1$ (Ojo [6]) for the ordinary logistic model.

Thus in the analysis of bioassay and quantal response data, if model (1.0.0) is used, what Berkson's logit transform does for the ordinary logistic can be done for the model (1.0.0) by the transformation (2.5.1).

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