To Switch or Not To Switch?

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Abstract

This note shows that when the Bayes formula and certain game theory concepts are appropriately applied, the posterior probability of winning a prize in the famous Monty Hall game is not the counter-intuitive answer of 2/3 but it is ½. The moral of this exercise is that the Bayes formula should not be applied mechanically. Its application is both art and science. As in the case of equilibrium refinements in game theory, application of the Bayes formula requires both working backward and forward through the game tree.

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Consider the famous TV game show "Let's make a Deal," in which a contestant is given a choice of three closed doors. Behind one of the doors is an attractive prize, say, a car, and the other two doors are empty. After the contestant has picked a door, say, Door 1, which remains closed, the host Mr. Monty Hall, who knows which door hides the prize, opens one of the remaining two doors, say, Door 3, to reveal an empty door. The host then offers the contestant the option of switching to Door 2. Should the contestant stick to Door 1 or switch to Door 2?

This famous Monty Hall game problem, also known as the three-door game problem or the car-goat problem, was the focus of much discussion about a decade ago (see, e.g., vos Savant 1990a, 1990b, 1991a, 1991b, Morgan *et al.* 1991b, 1991c, Seymann 1991, Tierney 1991, among others) because of its underlying probability paradox. The objective of this paper is to argue that the conditional probability of winning is not 2/3, as some studies have argued or demonstrated. When both the Bayes formula and certain game theory concepts (see, e.g., Gintis 2000 for more details about these concepts) are taken into consideration, the conditional probability of winning is ½, which immediately appeals to the intuition of most people including many statisticians and mathematicians.

Many solutions to the Monty Hall problem have been offered, and they vary depending on the assumptions made (see Falk 1993, pp. 184-9, Gillman 1992, Isaac 1995, pp. 8-10 and pp. 26-7, Morgan *et al.* 1991a, Selvin 1975a, 1975b, and vos Savant 1990a, 1990b, 1991a). To avoid unnecessary confusion and potential controversies, the analysis below makes explicitly the following assumptions: (i) the host always offers to the contestant the option to switch or not after the contestant has chosen a door and the host has opened a door showing no prize behind it; (ii) the host does not have biased preferences in favour of which door to open; (iii) the host does not use any psychological factor or strategic behaviour to influence the contestant's choice to switch or not; and finally (iv) when the contestant (host) has no reason or information to favour one door over any other doors, he will choose at random (i.e., a uniform distribution is assigned to all equally likely outcomes). The counterintuitive solution that the contestant should always switch because the probability of winning the prize is 2/3 is based on applying the Bayes formula to find the conditional probability of winning. For exposition, let us re-derive the result here. First, the probability of having the prize behind Door i is P($A=i$) = 1/3, for i = 1, 2, and 3, because of equal priors. Here $A=i$ represents the event that the prize is behind Door i. Assume the contestant picks Door 1. Now let H=2 denote the event that the host opens Door i. By the rule of the game that the host cannot open a door to reveal the prize, the probability of having the host to open Door 2 when the prize is behind Door 2 is P(H=2 | A=2) = 0, whereas P(H=3 | A=2) = 1. By the same token, P(H=3 | A=3) = 0, and P(H=2 | A=3) = 1. But if the prize is behind Door 1, then by assumptions (ii)-(iv) above, P(H=2 | A=1) = P(H=3 | A=1) = ½. Suppose the host opens Door 3. Then by the Bayes formula, the posterior probability that the prize is behind Door 1 is given as:

$$P(A=1|H=3) = \frac{P(H=3|A=1)\;P(A=1)}{\sum_{i=1}^{3} P(H=3|A=i)\;P(A=i)} \qquad (1)$$

$$= \frac{1/2\times 1/3}{1/2\times 1/3 + 1\times 1/3 + 0\times 1/3} = 1/3.$$

Also P(A=2 | H =3) = 2/3 and P(A=3 | H= 3) = 0. Therefore, the contestant should switch to Door 2 because there is a higher posterior probability of winning the prize by doing so.

However, in the above analysis the posterior probabilities are computed by applying the Bayes formula without taking into consideration of the rules of the game when the relevant probabilities are assigned. Consider the game tree depicted in Figure 1, in which the relevant part of the game in extensive form and the probabilities used in Equation (1) are shown. Equation (1) is computed in a backward-looking manner. That is, the contestant knows that she has now reached one of the nodes C_4-C_7 within the non-singleton information set defined by the dotted line, where she is about to make the switching decision; so she looks back at the game tree to find out the relevant paths and probabilities so as to update the posterior probability of winning the prize. This is why $P(H=2 \mid A=1) = P(H=3 \mid A=1) = ½$ are used in the calculation. The application of the Bayes formula in such a backward-looking manner is usually appropriate in most situations, such as the typical urn problem (see, e.g., Mood, Graybill and Boes 1974, pp. 37-8).

However, a similar application may not be appropriate in this Monty Hall game because the assigned probabilities $P(H=2 \mid A=1) = P(H=3 \mid A=1) = ½$ are inconsistent with the rules of the game. *Ex ante*, i.e., before the host opens a door, it is true that $P(H=2 \mid A=1) = P(H=3 \mid A=1) = ½$ because, by our assumptions, he can simply toss a coin to decide whether to open Door 2 or Door 3. *Ex post*, the posterior probabilities should no longer be ½ after one of the two doors is opened. When Door 3 is opened, a forward-looking contestant knows that the path following C_4 is no longer admissible by the rules of the game and not what the host wants her to choose, simply because she has no choice of switching. Moreover, even if she was allowed to switch, switching to Door 3 would be irrational as it is already known to be empty. By looking forward she knows that she cannot be worse off in terms of the expected payoff by choosing the path following C_5 rather than C_4 to continue the game. This is so no matter whether Door

1 or Door 2 hides the prize and also whether she switches or not, again simply because Door 3 is already known to be empty. More formally, the expected payoff from choosing C_4 is given as $EV(C_4)$ = P(winning upon not switching) × payoff behind Door 1 + [1- P(winning upon not switching)] × 0, as the payoff behind Door 3 is already known to be zero. On the other hand, the expected payoff from choosing C_5 is $EV(C_5)$ = P(winning upon not switching) × payoff behind Door 1 + [1- P(winning upon not switching)] × payoff behind Door 2. Apparently, $EV(C_5) \geq EV(C_4)$ no matter what the value of P(winning upon no switching) is and which door hides the prize. Thus in the subgame after H_2 that consists of C_4 and C_5, a rational contestant will not choose the path following C_4 because doing so is not a best response for herself. Put differently, the contestant's strategies after C_4 are weakly dominated and, just like C_7, would not be chosen. Under this assessment, the assigned probabilities should be $P(H=2 \mid A=1) = 0$ and $P(H=3 \mid A=1) = 1$. Thus it follows that the posterior probability of winning the prize becomes

$$
\begin{aligned}
P(A=1|H=3) &= \frac{P(H=3|A=1)\, P(A=1)}{\sum_{i=1}^{3} P(H=3|A=i)\, P(A=i)} \qquad (2)\\
&= \frac{1 \times 1/3}{1 \times 1/3 + 1 \times 1/3 + 0 \times 1/3} = 1/2.
\end{aligned}
$$

Consequently, the contestant should be indifferent to switching or not.

It is recognized in the game theory literature that in certain cases a player's posterior belief cannot be calculated using the Bayes formula. More important, there is no fixed algorithm to update the posterior belief so long as the belief does not violate the Bayes Theorem. The analysis here demonstrates that the Bayes formula should not be applied mechanically. Its application is both art and science. In our case, a synthesis of probability theory, game theory, and simple logic gives us a "reasonable" and "sensible" posterior belief. Let us consider the puzzle in an alternative, simple way: given the rules of the Monty Hall game, the game is essentially a two-stage dynamic game with asymmetric information. In the first stage, the contestant's and the host's choices determine which two doors form the lottery for the next stage. But as far as the chance of winning the lottery is concerned, which two doors are selected is really irrelevant because by the rules of the game the two selected doors always consist of one with the prize behind it and the other without. While the form or appearance of the lottery to follow in the next stage can differ (i.e., the lottery can be Door 1 and Door 3, or Door 1 and Door 2, or any one of the other possible combinations), the underlying probability of winning the prize is always the same, i.e. ½. Monty Hall could have simply tossed a coin to decide whether the prize should go to the contestant or not. But doing so would make the TV show less exciting and interesting than it is.

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Figure 1: The Monty Hall Game as a Dynamic Game with Asymmetric Information

[Put Figure 1 here]

Explanatory Notes to Figure 1:
(1) At node H_1 the host picks a door to hide the prize.
(2) D_1, D_2 and D_3 denote respectively Doors 1, 2 and 3.
(3) The numbers are the related probabilities.
(4) At nodes C_1, C_2 and C_3, the contestant picks a door, which is Door 1 in our illustrative example.
(5) At nodes H_2, H_3 and H_4, the hosts opens a door. In our example, he opens Door 3.
(6) In the final stage the contestant reaches nodes C_4 - C_7, with the information set shown by the dotted line, and has to decide to switch or not.

Figure 1: The Monty Hall Game