ON A GENERALIZATION OF THE GUMBEL DISTRIBUTION

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Abstract

A simple generalization of the Gumbel distribution is proposed. Simple properties of the distribution are studied.

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1 INTRODUCTION

The importance of the Gumbel law in modelling stochastic environmental phenomena has been documented Gumbel [3]. The Gumbel distribution was obtained as a limiting distribution of the $m-th$ largest extremes.
There have been efforts at generalizing the distribution. Ojo [5] and Adeyemi and Ojo [1] have obtained different generalizations to the distribution. Ojo [5] also obtained some properties relating the generalized Gumbel distribution to some other distributions.

In these generalizations (Ojo [5], Adeyemi and Ojo [1], the cumulative distribution functions (cdf's) have no close forms.
In this paper, a simple generalization of the Gumbel distribution with close cdf is proposed. Some of its properties are investigated.

2 THE GENERALIZATION

The Gumbel probability distribution function is simply defined by

$$f(x) = e^{-x} exp(-e^{-x}), \quad -\infty \; < \; x \; < \; \infty \tag{2.1}$$

and the cumulative disribution function is given as

$$F(x) = exp(-e^{-x}) \tag{2.2}$$

By introducing a shape parameter $\lambda > 0$, the random variable Y will be called the generalized Gumbel random variable if its density function is written as

$$g(y) = \lambda e^{-\lambda y} exp(-e^{-\lambda y}), \quad -\infty \; < \; x \; < \; \infty \tag{2.3}$$

and the cumulative distribution function is

$$G(y) = e^{-e^{-\lambda y}} \tag{2.4}$$

The distribution (2.3) is proposed as a generalization of the Gumbel distribution such that when $\lambda = 1$, the distribution (2.3) reduces to (2.1).

Remark The distribution is more flexible than the ordinary Gumbel density. It is easily observed that probability evaluation from (2.4) is simple. The pdf and the cdf are related by

$$\ln g(y) - \ln G(y) + \lambda y = 0$$

The characteristic function of the generalized version (2.3) is given as

$$\Phi_Y(t) = \lambda \int_{-\infty}^{\infty} e^{ity} e^{-\lambda y} exp(-e^{-\lambda y}) dy$$

$$= \int_0^\infty u^{-\frac{it}{\lambda}} e^{-u} du$$

$$= \Gamma(1 - \frac{it}{\lambda}) \tag{2.5}$$

Its corresponding moment generating function is given as

$$M_Y(t) = \Gamma(1 - \frac{t}{\lambda})$$

3 THE CUMULANTS

By adopting the method of Adeyemi and Ojo [1], using the characteristic function (2.5) and a known definition of the digamma function we hereby derive the cumulants for positive integer values of λ thus the cumulant generating function is given by

$$C_Y(t) = log_e \Phi(t)$$

and the r^{th} cumulants are obtained by

$$\kappa_r = \frac{d^r}{dt^r} C_Y(t)|_{t=0}$$

$$= \frac{d^r}{dt^r} log_e \Gamma(1 - \frac{it}{\lambda})|_{t=0} \tag{3.1}$$

By using

$$\frac{d^r}{dt^r} log_e \Gamma(z+1) = \Phi^{(r-1)}(z) = (r-1)!(-1)^r \Sigma_{j=0}^{\infty} (z+j)^{-r} \tag{3.2}$$

we have the cumulants as

$$\kappa_r = (\frac{1}{\lambda})^r (r-1)! \Sigma_{j=0}^{\infty} (1+j)^{-r} \tag{3.3}$$

Thus,after simplification, the first four cumulants for integer λ are

$$\kappa_1 = \mu = \frac{1}{\lambda}[log_e \lambda + \gamma - \Sigma_{\lambda=0}^{\infty} \lambda^{-1}] \tag{3.4}$$

$$\kappa_2 = \sigma^2 = (\frac{\pi}{\sqrt{6}\lambda})^2 \tag{3.5}$$

$$\kappa_3 = \frac{2}{\lambda^3} \Sigma_{j=1}^{\infty} j^{-3} \tag{3.6}$$

$$\kappa_4 = \frac{\pi^4}{15\lambda^4} \tag{3.7}$$

For $\lambda = m + \frac{1}{2}$, where m is a positive integer we have from (3.3)

$$\kappa_r = (\frac{2}{2m+1})^r (r-1)! \Sigma_{j=0}^{\infty} (1+j)^{-r} \tag{3.8}$$

and the first four cumulants becomes

$$\kappa_1 = \frac{2}{(2m+1)} \Sigma_{j=0}^{\infty} j^{-1} \tag{3.9}$$

$$\kappa_2 = \frac{1}{6}(\frac{2\pi}{2m+1})^2 \tag{3.10}$$

$$\kappa_3 = (\frac{2}{2m+1})^3 \Sigma_{j=1}^{\infty} j^{-3} \tag{3.11}$$

$$\kappa_4 = \frac{1}{15}(\frac{2\pi}{2m+1})^4 \tag{3.12}$$

For general values of λ, we shall use a different procedure to obtain good approximations to the cumulants. Let us apply the following Stirling's approximation to the gamma function

$$log_e \Gamma(z) \simeq \frac{1}{2} log_e 2\pi + (z - \frac{1}{2}) log_e z - z + \frac{1}{12z}$$

Therefore,

$$log_e M_y(t) = log_e \Gamma(1 - \frac{t}{\lambda})$$

$$= \frac{1}{2} log_e 2\pi + (\frac{1}{2} - \frac{t}{\lambda}) log_e (1 - \frac{t}{\lambda}) - (1 - \frac{t}{\lambda}) + \frac{1}{12(1 - \frac{t}{\lambda})} \tag{3.13}$$

By differentiating repeatedly with respect to t and putting $t = 0$, we have the first four cumulants as

$$\kappa_1 \simeq \frac{7}{12\lambda} \tag{3.14}$$

$$\kappa_2 \simeq \frac{10}{6\lambda^2} \tag{3.15}$$

$$\kappa_3 \simeq \frac{5}{\lambda^3} \tag{3.16}$$

$$\kappa_4 \simeq \frac{7}{\lambda^4} \tag{3.17}$$

Remark It would be observed that the above is a good results for approximating cumulants for the proposed generalized Gumbel distribution for various values of λ. The coefficient of kurtosis, $\beta_2(y)$ is constant $\approx$2.4. The distribution is skewed to the right, since the coefficient of skewness, $\beta_1 > 0$ for $\lambda > 0$.

4 CHARACTERIZATION THEOREMS

Some characterization theorems are hereby proved.

Theorem 4.1 Let X be a continuous random variable with density function $g(x)$. Then the random variable $Y = -\frac{1}{\lambda}log_e X$ has a generalized Gumbel distribution proposed in (2.3) if and only if X has an exponential distribution.

Proof Suppose X has an exponential distribution

$$g(x) = e^{-x}, \quad 0 < x < \infty$$

It is not difficult to show by change of variable technique that the random variable $Y = -\frac{1}{\lambda}log_e X$ is generalized Gumbel.

Conversely, suppose $Y = -\frac{1}{\lambda}log_e X$ is generalised Gumbel. Then the moment generating function $M_Y(t)$ of Y is given by

$$M_Y(t) = \Gamma(1 - \frac{t}{\lambda})$$

If

$$M_{-\frac{1}{\lambda}log_e X(t) = \Gamma(1-\frac{t}{\lambda}})$$

then

$$E[e^{-\frac{1}{\lambda}log_e x}t] = E[x^{-\frac{t}{\lambda}}]$$

$$= \int_0^\infty x^{-\frac{t}{\lambda}} f(x)dx$$

$$= \Gamma(1 - \frac{t}{\lambda})$$

if and only if $f(x)$ is an exponential distribution.

Theorem 4.2 Let X be a continuous random variable with density function $h(x)$. Then the random variable $Y = -\frac{1}{\lambda}\ln(-\ln X)$ has a generalized Gumbel distribution if X has a uniform distribution.

Proof Suppose X has a uniform distribution

$$h(x) = a \quad , 0 \leq a \leq 1$$

$$= 0 \quad , otherwise$$

By change of variable technique, it is easy to see that Y is a generalized Gumbel distribution.

Theorem 4.3 The random variable Y is generalized Gumbel with parameter λ if and only if it's probability distribution function satisfies the first order homogeneous differential equation

$$\lambda[e^{-\lambda y} - 1]f - f' = 0 \tag{4.3.1}$$

(prime denotes differentiation and f is written for $f(y)$)

Proof Suppose Y is generalized Gumbel, the density function of Y is given by (2.3)Differentiating (2.3) with respect to y and substituting into (4.3.1) gives the proof of the first part.

Conversely, suppose the density function f satisfies (4.3.1) , we have

$$\lambda[e^{-\lambda y} - 1]f = \frac{df}{dy}$$

Seperating variables and integrating, we have the solution of the differentaial equation (4.2.1) as

$$f = ke^{-\lambda y}e^{-e^{-\lambda y}}$$

Since f is a density function,

$$\int_{-\infty}^{\infty} f dy = 1$$

Direct integration gives $k = \lambda$, and thus the proof is complete.

Theorem 4.3 The random variable Y is generalized Gumbel with parameter λ if it's cumulative distribution function satisfies the first order homogeneous differential equation

$$\ln F' - \ln F + \lambda y = 0 \tag{4.4.1}$$

(prime denotes differentiation and F is written for $F(y)$

Proof Suppose Y is generalized Gumbel, cdf of Y is given by (2.4). Differentiating (2.4) with respect to y and substituting into (4.4.1) gives the proof.

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