ON SMOOTHING TIME SERIES DATA USING A CLASSICAL MOVING AVERAGE FORMULA

Khogali A. Khogali , J. W Odhiambo and John. O. Owino.

Mathematics Department, University of Nairobi, P.O.Box 30197, Nairobi, Kenya.

Email: khogh@hotmail.com, Jowino22@yahoo.com or mathuon@icconet.ke

Abstract

In time series realizations, assuming that the trend component to be approximated by a polynomial in time, smoothing filters based on a moving-average formula are proposed which link the degree of this polynomial to the number of terms operating on the moving-average formula. Their properties are examined. Illustrations via data collected from three East Africa Regional meteorological stations are reported.

Key Words: Trend, Linear Time Invariant Moving-average Filter, Differencing and Unbiased Estimator.

1.Introduction

Smoothing procedures are descriptive rather than analytical, Anderson (1971). Once an error model is postulated, the problem is either to estimate or eliminate the trend m_t, the

systematic component of this model. With $(n+1)$ observations $y_1, y_2, \ldots, y_{n+1}$, we may decompose each of them according to the following classical model

$$y_t = m_t + s_t + u_t \tag{1.1}$$

where, u_t is the random error component, assumed to be covariance stationary, for instance u_t may be measurement errors, s_t is the seasonal or periodic component and m_t is the trend component which in turn may include a cyclical component. We assume that m_t can be approximated by a polynomial in time of degree q, that is

$$m_t = f(t) = \sum_{i=0}^{q} a_i t^i \tag{1.2}$$

The moving-average estimator $m_t^\star$ of m_t can be defined in the form:

$$E(m_t^\star) = \sum_{s=-\infty}^{\infty} c_s m_{t+s} \tag{1.3}$$

where, $E(.)$ denotes the expected value operator. In practice, we usually assume that the remote observations do not greatly contribute in estimating m_t so that for a positive integer h, we can set $c_s = 0$ for values where $h \leq |s|$, accordingly (1.3) reduces to:

$$E(m_t^\star) = \sum_{s=-h}^{h} c_s m_{t+s} \quad t = h+1,\, h+2,\, \ldots,\, n-h \tag{1.4}$$

Alternatively, by viewing the moving-average as a linear combination of the observations with coefficients that can be tabulated in standard form see Arnold (1992), we can express $m_t^\star$ as:

$$m_t^\star = \sum_{s=-h}^{h} c_s y_{t+s} \quad t = h+1,\, h+2,\, \ldots,\, n-h \tag{1.5}$$

When $c_s = c_{-s}$, we refer to it as a symmetric moving -average filter. Several methods are proposed in the literature to fit a trend using a moving-average formula, by specifying or obtaining estimates for the weights (c_s). Earlier studies for fitting m_t include both the Logistic Curve and the Compertz Curve. Moreover orthogonal polynomials are proposed since the mid of the last century, see for instance Anderson and Houseman (1942). Weighted averages or the Least Squars Method, LSM is of common use, see for instance Kendall and Stuart (1968). Anderson (1971) referred to LSM as polynomial smoothing. When $c_s = (2h+1)^{-1}$, then (1.5) defines the simple moving-average. Chatfield (1987) described both the geometric scheme of weights as well as the scheme based on the successive terms in the expansion of $(0.5+0.5)^{2h}$. Iterative average as a general form of moving -average are also employed, for a detailed discussion see Kendall and Stuart (1968) who showed that Spencer -moving average formulas are special cases of iterative averages.

However all those mentioned methods and procedures for estimating m_t are sharing one drawback, since by the nature of their construction they can not estimate the end effects of m_t for a given time series realization, unless a supplementary procedure is used, although the asymmetric formulas reduce the magnitude of this problem. These formulas are given by:

$$m^{\star}_{1t} = \sum_{s=0}^{h} c_s y_{t+s} \quad t = h+1,\, h+2,\, \ldots,\, n-h \tag{1.6}$$

and

$$m^{\star}_{2t} = \sum_{s=0}^{h} c_s y_{t-s} \quad t = h+1,\, h+2,\, \ldots,\, n-h \tag{1.7}$$

respectively. Fuller (1976), referred to (1.7), for instance as the left side moving average, a terminology we adopt in this study. This article is based on theoretical results developed by Khogali And Odhiambo (1998) who introduced a filter; the Trend Estimation Method TEM as an alternative trend smoothing procedure to the famous Least squared Method, LSM which is already in standard use for smoothing crude series in which a polynomial trend component is assumed to exist, see for instance the$SPSS$ soft package in which LSM is of a common use in the social, economical, physical and biological applications. Khogali (1998) who adjust TEM to smooth some Bio-data taken from a Kenyan Meteorological stations, found that LSM and TEM perform invariably and Khogali, Odhiambo and Owino(2001) who compared LSM and TEM by using regional data from east Africa . These Theoretical results provide an unbiased estimates for the trend m_t , by obtaining tabulated values for (c_s) in (1.5),(1.6) and (1.7) respectively. These weights are then combined to allow for m_t to pass for each of the $n+1$ observations. Accordingly in section 2 we present the proposed filters. Section 3 investigates on some of their properties. Section 4 provides an evidence, a new development, that shows generally TEM out-perform LSM. Illustrations using data extracted from three East African meteorological stations are placed in section 5. A Conclusion is drawn in section 6.

2.The Proposed Filters

We present the following unbiased, linear and time invariant filters

(a) The Left Side Moving-average Formula

With $\Delta = 1 - L$ being the differencing operator, where L is the lag operator such that $Lm_t = m_{t-1}$, it is has already been established in theory, see Anderson (1971) that

$$\Delta^r m_t = \Delta^r f(t) = 0, \text{ for } r > q \tag{2.1}$$

Now, in view of (1.1) we may use (2.1) to eliminate the trend m_t and if we subtract the remaining components for each observation from its corresponding crude observation, we may return back with an estimate for m_t Symbolically, removing the trend will result in

$$\Delta^r y_t = \Delta^r (s_t + u_t) \tag{2.2}$$

and by subtracting those remaining components from y_t, we obtain

$$m^\star_{2t} = y_t - \Delta^r (s_t + u_t) = y_t - \Delta^r y_t \tag{2.3}$$

Define,

$$d_j = \begin{cases} 1 & j = 0 \\ 0 & j \neq 0 \end{cases}$$

so that, we can rewrite (2.3) with regard to (1.7) as:

$$\sum_{s=0}^{h} c_s y_{t-s} = \sum_{j=0}^{r} \{d_j + (-1)^{j+1} \binom{r}{j} \} y_{t-j} \tag{2.4}$$

where $\binom{r}{j} = r!/j!\,(r-j)!$. If we set $r = h = q + w$, $w = 1, 2, \ldots$ then one of the solutions of (2.4) is to equate the coefficients of y_{t-j}, immediately, we obtain :

$$c_j = \begin{cases} 0 & j = 0 \\ (-1)^{j+1} \binom{r}{j} & j = 1, 2, \ldots, r \end{cases} \tag{2.5}$$

Setting $r = h$ is a sufficient and necessary restriction for uniquness of this solution see Khogali (1997). This filter as we will show in the next section is linear, time variant and unbiased filter.

Since c_s are alternating in sign, then we may estimate one or more values as negative quantities which may contradict the definition of some series that are strictly assumed to be nonnegative. Accordingly we propose the following $\{c_s^{\star}\}$ normalized filter which works in absolute terms

$$c_s^{\star} = |c_s|\{\sum_{j=1}^{q+1} |c_j|\}^{-1} \qquad s = 0,\, 1,\, \ldots,\, q+1 \tag{2.6}$$

This filter as we will see in the next section is asymptotically unbiased, linear and time invariant filter.

It deserves to note that Anderson (1971) is pioneer in linking the degree of the polynomial assumed to estimate the trend with the number of terms operating in the moving average formula. Anderson (1971) considered the case when $h = [q]/2$, where $[k] = k$ when k is even and $[k] = k - 1$ when k is odd. Accordingly by applying LSM, Anderson (1971) obtained the following results:

$$C_s = C_{-s} = (-1)^{s+1} C \binom{2k+2}{k+1+s}; C_0 = 1 - C\binom{2k+2}{k+1}$$

where

$$C = \binom{2k+2}{k+1} \Big/ \binom{4k+4}{2k+2}$$

(b) The Right Side Moving-average Formula

This formula according to Fuller (1976), is defined as:

$$E(m_t^\star) = \sum_{s=0}^{h} r_s m_{t+s} \tag{2.7}$$

However, we may relate (2.7) to $m_{2t}^\star$ in the following manner:

$$E(m_{2t}^\star) = \sum_{s=0}^{h} c_s m_{t-s} = \sum_{s=1}^{h-1} \delta_h c_{h-s} m_{t-s} + c_0 m_t + c_h m_{t-h}$$

where, $c_s = \delta_h c_{h-s}$, $(s = 1,\, 2,\, \ldots,\, h-1)$ and $\delta_h = 1$ or -1 depending on whether h is an even number or odd number respectively as we will show in the next section. Further more with $r_{h-s} = \delta_h c_{h-s} = c_s$ and $t' = t - h$ we may write:

$$\begin{aligned} E(m_{2t}^\star) &= \textstyle\sum_{s=1}^{h-1} r_{h-s} m_{t-s} + r_h m_t + r_0 m_{t-h} \\ &= \textstyle\sum_{j=0}^{h} r_j m_{t'+j} \\ &= E(m_{1t'}^\star) \end{aligned} \tag{2.8}$$

Thus, this manipulation allow us to relate the left side moving average operating at time t to the right side moving formula operating at period $t' = t - h$. Thus:

$$r_j = \delta_h c_{h-j} = \begin{cases} c_j & h:\ \text{even} \qquad j = 1,\, 2,\, \ldots,\, h-1 \\ -c_j & h:\ \text{odd} \qquad j = 1,\, 2,\, \ldots,\, h-1 \\ (-1)^{h+1} & j = 0 \\ 0 & j = h \end{cases} \tag{2.9}$$

(c) The Symmetric Moving-average

Which is already defined by (1.4) may be split into the following two terms:

$$E(m_t^\star) = \sum_{s=-h}^{h} a_s m_{t+s} = \sum_{s=0}^{h} a_s m_{t+s} + \sum_{s=1}^{h} a_{-s} m_{t-s} \tag{2.10}$$

Due to the symmetric property of the weights of this filter and in view of (2.10) that relates these weights to each of the one sided formulas in a manner as if the estimates of m_t has been done twice by either of the one sided moving- averages. If we regard in addition the relationship between (c_s) and (r_s),then it is logical to set $E(m_t^\star) = 0.5E(m_{2t}^\star) = 0.5E(m_{1t}^\star)$. Formally assuming $a_s = c_s = r_0, 1 \leq s \leq h-1$, $r_0 = c_h$ and $r_h = 0 = c_0$ then (2.10) gives

$$E(m_t^\star) = E(m_{2t}^\star) + E(m_{1t}^\star) = m_t + m_t = 2m_t \tag{2.11}$$

Accordingly adjusting for unbiasedness we get

$$a_j = a_{-j} = \begin{cases} 0 & j = 0 \\ 0.5c_j & j = 1,\, 2,\, \ldots,\, h \end{cases}$$

and we note that like the Left Side Moving formula, the central observation y_t has a zero weight. Now it is evident from relationship between (c_s), (r_s) and (a_s) the end effects may be estimated by using any appropriate combinations of these weights.

3.Properties of the Filters

(i) Linear Time Invariant Filters

By their definitions, each of these filters is linear and time invariant filter such that $\sum_s c_s = \sum_s r_s = \sum_s a_s = 1$. Moreover due to the fact that for $0 < r < n$, we have

$\binom{n}{r} = \binom{n}{n-r}$ then we can write:

$$c_{h-j} = \begin{cases} (-1)^{h-j+1}\binom{h}{h-j} = c_j & h:\text{even} j = 1,\, 2,\, \ldots,\, h-1 \\ (-1)^{h-j+1}\binom{h}{h-j} = -c_j & h:\text{odd} j = 1,\, 2,\, \ldots,\, h-1 \\ (-1)^{h+1} & j = 0 \\ 0 & j = h \end{cases} \tag{3.1}$$

which we have used in (2.8) to derive (2.9).

(ii) Unbiased Filters

They are unbiased in the sense that, they allow the trend m_t to pass through them, once the degree of the polynomial assumed to approximate m_t is estimated, for it's estimation see Anderson (1971). To show this we consider the following:

For the Left Side Moving-average:

To show that $E(m^{\star}_{2t}) = m_t$, we obtain after few omitted intermediate steps:

$$\begin{aligned} E(m^{\star}_{2t}) &= \textstyle\sum_{s=0}^{h} c_s m_{t-s} \\ &= m_t - \Delta^h m_t \\ &= m_t \;\; \text{since} \;\; h > q \end{aligned} \tag{3.2}$$

see (Appendix A).

On the other hand considering $\{c^{\star}_s\}$ we obtain the same result in a symptotic sense, since after few omitted steps, see Appendix B

$$E(m^{\star}_{2t}) = \sum_{s=0}^{q+1} c^{\star}_s m_{t-s} = m_t \;\; \text{since} \;\; q+1 > q \tag{3.3}$$

For the Right Side Moving average

In a similar computations, to show that $E(m^{\star}_{1t'}) = m'_t$, assuming that $L^0 = 1$, we obtain after few intermediate steps, see Appendix C we get:

$$\begin{aligned} E(m^{\star}_{1t'}) &= \textstyle\sum_{s=0}^{h} c_s m_{t'+s} \\ &= \textstyle\sum_{s=0}^{h} c_s L^{-s} m_{t'} \\ &= -\Delta^h m_{t'+h} + m_{t'+h} \\ &= m_t \ \text{ since } \ h > q \end{aligned}$$

For the symmetric formula, the result is readily obtainable from either directly or from the definition of (a_s) in connection to the asymmetric formulas.

(iii) The Arbitrariness in deciding the number of terms in the Moving-average Formula

Unlike most of the other filters, these smoothing procedures are free from the arbitrariness in deciding h, since each of them links those terms to the degree k, of the polynomial that assume to approximate m_t.

(iv) Decomosition of the trend m_t

When the trend m_t is a sum of a linear component $m_t^{(1)} = a + bt$ and a nonlinear component of the form $m_t^{(2)} = cd^t$ and $d = 2 \, or \, 1/2$, then applying the proposed formulas will result in estimating the linear component m_t alone. This can be verified for the left side formula with $\{c_0 = 0,\, c_1 = 2,\, c_2 = -1\}$ and $d = 1/2$, noting that $m_{t-1}^{(1)} = \frac{m_t^{(1)} + m_{t-2}^{(1)}}{2}$ as follows:

$$\begin{aligned} E(m_t) &= \textstyle\sum_{i=0}^{2} c_i m_{t-i}^{(1)} + \sum_{i=0}^{2} c_i m_{t-i}^{(2)} \\ &= 2m_{t-1}^{(1)} - m_{t-2}^{(1)} + c(1/2)^{(t-2)} - c(1/2)^{(t-2)} \\ &= m_t^{(1)} \end{aligned} \tag{3.5}$$

Due to the integer nature of the proposed weights, other specific patterrns of the trend m_t may be decomposed.

(v) Handling of missing data

Again, unlike, the other filters since $a_0 = c_0 = r_h = 0$, then those proposed filters have the advantage of considering missing observations of various patterns. For instance, if the n-th observation in each consecutive n observations are missing, then by locating them to correspond to either c_0 or a_0 we may obtain estimates for their respective trend component. Further more since, adding extra terms would not affect the estimates of m_t, then the appropriate choice of ω will ensure that the location of the zeros against those missing values can be easily achieved.

4.The Biasness in Estimating the trend Component

Anderson(1971) noted that for a given $k = [q]/2$, the variance of the estimated trend is inversely related to the terms m in which the filter is based. On the other hand Anderson (1971) also noted that for a given m then k is propertionally related to the bias in estimating the trend $\hat{m}_t$ accordingly he proposed the Mean Squared Error MSE rather than the variance, as a criterin to assess the extend of smoothing. In this section it can be shown that for a given k TEM will be based on $m = m_1$ terms greater than the number of terms $m = m_2$ for the corresponding LSM. Consequently aplying the MSE criterion we claim that for a given $[q]$ or $k = [q]/2$ then it is the variance of the estimated m_t generated by TEM has a variance relatively less than the variance of m_t generated

by LSM.

Formally, Anderson (1971) consider the case $m = k = [q]/2$, construct the variance of m_t as:

$$Var(\hat{m_t}) = \sigma^2\{1 - \frac{(2k+2)!^4}{(4k+4)!(k+1)!^4}\} \tag{4.1}$$

When $k = 0$ so that $q = 0$ or $q = 1$, then $Var(\hat{m_t}) = \sigma^2/3$. With $k = 1$ so that m_t is either a quadratic or cubic, equation (4.1) reduces to

$$Var(\hat{m_t}) = \frac{3(3m^2+3m-1)}{(2m-1)(2m+1)(2m+3)}\sigma^2 \tag{4.2}$$

So with $m = 2$, $Var(\hat{m_t}) = 17/13\sigma^2$see Table 4.1

Table4.1: The Variances of the Estimated Trend Assuming $\sigma^2 = 1$

q	$k = [q]/2$	LSM	$2m_2 + 1 = [q] + 3$	TEM	$2m_1 + 1 = 2[q] + 3$
0	0	0.3333	3	0.3333	3
1	0	0.3333	3	0.2778	5
2	1	0.4857	5	0.1939	7
3	1	0.4857	5	0.1533	9
4	2	0.5670	7	0.0769	11
5	2	0.5670	7	0.0870	13
6	3	0.6190	9	0.1064	15
7	3	0.6190	9	0.0852	17

Now with regard to TEM, since $\Sigma_{j=0}^{n}\binom{n}{j} = (1+1)^n$ and $c_0 = 0$ then for the asymmetric

formula the variance of $\hat{m_{1t}}$ is given as

$$Var(\hat{m_{1t}}) = \frac{\Sigma_{j=0}^{k+1}\binom{k+1}{j}^2\sigma^2}{(2^{k+1}-1)^2} \tag{4.3}$$

and for the symmetric formula we have

$$Var(\hat{m_t}) = \frac{\Sigma_{j=0}^{k+1}\binom{k+1}{j}^2\sigma^2}{2(2^{k+1}-1)^2} \tag{4.4}$$

Using equation(4.4) we compute the variance for $k = 0, 1, \ldots 7$. from Table 4.1 it is quite evident that the variance of the estimated trend generated by TEM is less than the corresponding varance of the trend generated by LSM. This is may not be a surprised result if we note that TEM is based on $2[q] + 3$ terms while LSM is only based on $[q] + 3$ terms as shown in Table4.1. Indeed Table4.1 is self-explanatory. TEM provides the estimated variance for each q while LSM provides the same estimated variance for each concecutive two degrees.

5. Empirical Illustrations

With seasonal data for rainfall taken from three meteorological stations Lodwar, Masindi and Diredawa for the period 1961-1990, we estimated the trend component using LSM and TEM as reported in Table 5.1, Table 5.2 and Table 5.3 respectively

Table5.1: Smoothing Rainfall for Lodwar-Kenya (1961-1990)

$Year$	$Rain(mm)$	LSM	TEM	$Year$	$Rain(mm)$	LSM	TEM
1961	393.8		275.4	1976	7.5	78.5	76.9
1962	38.7		55.1	1977	217.9	110.18	15.35
1963	87.8	45.7	104.9	1978	16.3	93.96	90.6
1964	72.2	64.3	44.8	1979	40.0	13.25	48.62
1965	13.8	23.8	66.3	1980	20.3	1.63	56.6
1966	27.7	47.8	56.3	1981	0.6	83.07	101.43
1967	109.9	63.7	25.3	1982	242.1	129.94	20.38
1968	18.3	60.4	57.0	1983	43.8	108.4	85.72
1969	47.2	20.0	25.0	1984	13.2	1.23	56.98
1970	0.8	5.1	38.7	1985	3.1	2.41	17.43
1971	1.9	29.8	60.9	1986	6	3.34	24.67
1972	115.9	85.2	28.4	1987	22.4	33.42	45.97
1973	82.7	79.4	39.8	1988	83.8	77.89	43.20
1974	0.2	20.3	49.3	1989	93.1		63.33
1975	4.5	0	52.7	1990	22.2		90.00

With Standard error of 39.12 and 25.01 for LSM and TEM respectively.

Table5.2: Smoothing Rainfall for Masindi-Uganda (1961-1990)

$Year$	$Rain(mm)$	LSM	TEM	$Year$	$Rain(mm)$	LSM	TEM
1961	859.8		757.8	1976	370.7	423.3	523.89
1962	553.8		552.6	1977	597.3	619.89	529.89
1963	550.3	497.32	580.92	1978	799.9	672.1	473.4
1964	474.4	531.5	534.1	1979	427.7	547.0	560.1
1965	569.3	483.9	482.9	1980	419.9	357.1	453.7
1966	411.8	508.0	505.2	1981	323.7	372.9	419.7
1967	574.8	497.9	495.3	1982	419.3	387.3	378.6
1968	508.8	564.3	519.3	1983	412	406.1	399.3
1969	561.2	513.3	472.7	1984	380.5	408.9	448.9
1970	431.9	427.1	518.9	1985	472.7	447.7	461.3
1971	380.2	484.2	571.3	1986	504.8	538.3	487.7
1972	721.8	607.1	440.9	1987	585	512.9	490.3
1973	559.4	566.2	483.2	1988	430.5	667.6	592.9
1974	334	400.1	504.4	1989	598.7		482
1975	407.4	340.4	427.7	1990	685.4		542.6

With Standard error of 91.89 and 55.24 for LSM and TEM respectively.

Table5.3: Smoothing Rainfall for Diredawa-Ethiopia (1961-1990)

$Year$	$Rain(mm)$	LSM	TEM	$Year$	$Rain(mm)$	LSM	TEM
1961	255.5		215.1	1976	162.5	145.1	100.03
1962	134.2		129.9	1977	133.8	127.9	110.87
1963	121.4	138.4	162.8	1978	59	95.1	131.8
1964	183.1	143.8	101.7	1979	127.2	102.5	86.03
1965	86.5	90.2	148.9	1980	106.2	89.37	105.7
1966	59.1	130.7	176.3	1981	52	118.02	142.8
1967	287.4	192.6	80.5	1982	217	136.9	74.02
1968	124.5	155.9	130.5	1983	83.4	123.06	123.58
1969	28.7	55.1	136.4	1984	67.1	79.1	114.2
1970	92	74.9	68.2	1985	121.3	88.3	68.8
1971	98.3	81.7	56.6	1986	58.9	80.9	97.1
1972	30.7	55.9	83.8	1987	77.9	79.5	95.4
1973	65.4	54.4	70.3	1988	117.8	62.1	75.1
1974	83.6	79.04	85.7	1989	97.6		104.4
1975	95	130.4	115.2	1990	41.5		104.3

With Standard error of 35.27 and 31.46 for LSM and TEM respectively.

Please insert here Figure 1, figure 2 and Figure 3 respectively

Figures 1,2 and 3 are self explanatory in comparing the TEM method and LSM method in smoothing rainfall for the three selected countries in confirmity with tables 5.1, 5.2 and 5.3 respectively.

6.Concluding Remarks

In this article, a re-visit to the classical estimation of the trend by applying a moving - average formulas is presented. Regarding the complementary relation between trend elimination and trend estimation for a given time series realization, we propose a set of weights for linear time invariant moving - average formulas. These filters turn out to be unbiased with the advantage of linking the degree of the polynomial assumed to approximate the trend, to the number of terms operating in this moving formulas aswell as estimating the end effects which usually needed for forecasting. Indeed since smoothing is a descriptive technique, experience and intuition play their role in selecting among the competing procedures. Using real data we have illustrated that TEM performed better than LSM in smoothing three bio-data series as measured by the drastic reduction in the standard errors of those series. Moreover TEM has the advantage of estimating the end effects. Moreover we have seen that for a given polynomial degree, TEM is based in number of terms exceed those terms in whichg LSM is based this is may partially explain why for agiven realization the former filter perform better in smoothing than the later filter, see Table 4.1

Appendices

Appendix A

$$
\begin{aligned}
E(m^{\star}_{2t}) &= \textstyle\sum_{s=0}^{h} c_s m_{t-s} \\
&= \textstyle\sum_{s=0}^{h} c_s L^s m_t \\
&= \{\binom{h}{1} L + \binom{h}{2} L^2 + \ldots + (-1)^{(h+1)} L^h\} m_t \\
&= \{\pm 1 + \binom{h}{1}(-L) + \binom{h}{2}(-L)^2 + \ldots \binom{h}{h}(-L)^h\} m_t \\
&= m_t - \Delta^h m_t \\
&= m_t \ \text{ since } \ h > q
\end{aligned}
$$

Appendix B

$$
\begin{aligned}
\lim_{k\to\infty} E(m^{\star}_{2t} &= \lim_{k\to\infty} \textstyle\sum_{s=0}^{q+1} |c_s| L^s \{\sum_{j=1}^{q+1} |c_j|\}^{-1} \\
&= \lim_{k\to\infty} |c_s| L^s m_t 2^{-(k+1)} \\
&= \lim_{k\to\infty} \{(1+L)^{k+1} - 1\}\{2^{(k+1)} - 1\}^{(-1)} m_t \\
&= \lim_{k\to\infty} \{(2-\Delta/2)^{(k+1)} - 1\}\{2^{(k+1)} - 1\}^{(-1)} m_t \\
&= \lim_{k\to\infty} \{(1-\Delta/2)^{k+1} + 1\} m_t \\
&= m_t \ \text{ since } \ |\Delta/2| < 1
\end{aligned}
$$

Appendix C

$$
\begin{aligned}
E(m^{\star}_{1t'}) &= \textstyle\sum_{s=0}^{h} c_s m_{t'+s} \\
&= \textstyle\sum_{s=0}^{h} c_s L^{-s} m_{t'} \\
&= \{c_h L^h + c_{h-1} L^{h-1} + c_{h-2} L^{h-2} + \ldots c_0 L^0\} \\
&= (-1)^{h+1}\{\tbinom{h}{h} + (-1)^h \tbinom{h}{h-1} L^{h-1} + \ldots + (-1)^{(2)} \tbinom{h}{1} L\} m_{t'+h} \\
&= -\{\pm \tbinom{h}{h}(-L)^0 + \tbinom{h}{1}(-L)^{h-1} + \tbinom{h}{2}(-L)^{h-2} + \ldots + \tbinom{h}{h-1}(-L)^1\} m_{t'+h} \\
&= -\{(1-L)^h - 1\} m_{t'+h} \\
&= -\{\Delta^h - 1\} m_{t'+h} \\
&= -\Delta^h m_{t'+h} + m_{t'+h} \\
&= m_t \quad \text{since} \quad h > q
\end{aligned}
$$

Acknowledgement

The authors are gratefully aknowledge the Department of mathematics, University of Nairobi for providing the Computer facility to produce this work and to Dr. Olorunsola E. Olowofeso a Visiting lecturer from the Federal University of Technology, Akure, Ondo State, Nigeria for his help in editing this article in PCTeX 32

References

[1] Anderson, R. L. and Houseman, E. E. (1942) "Tables of orthogonal polynomial values to N=104 " *(Research Bulletin* ***297****), Agricultural Experiment Station, Iowa State College, Ames. Iowa.*

[2] Anderson, T. W. (1971)" The Statistical Analysis of Time Series "*John Wiley and Sons Inc. New York.*

[3] Arnold, E. (1992)" Time Series " *3rd ed. Co published in the Americas by Holsted Press, an imprint of John Wiley and Sons Inc. New York, Toronto.*

[4] Chatfield, C. (1987)" The Statistical Analysis of Time Series :An introduction "*3rd ed. Chapman and Hall.*

[5] Fuller, W. A. (1976) " Introduction to Statistical Time Series "*John Wiley, New York.*

[6] Kendall, M.G. and Stuart, A. (1968) "The Advance Theory of Statistics "*Vol. Griffin, London.*

[7] Khogali, A. K. (1997) "On Econometric Modelling and Time series Models"*Ph.D thesis, University of Nairobi, Nairobi.*

[8] Khogali, A. K. and Odhiambo, J. W. (1998) "On the Classical Estimation of the Trend using a Moving-average Formula *A Paper presented at the Third Conference on Econometrics, 10-12 July 1998 hosted by department of Economics University of Pretoria, South Africa.*

[9] Khogali, A. K. (1998) "On Smoothing Bio-Data using a Classical Estimation Method "*An article presented at the Fourth International Statistic Conference,23-25 August 1998 hosted by Shahid Beheshti University, Iran.*

[10] Khogali, A. K., J. W. Odhiambo and J. W. Owino (2001) “On Smoothing Climatological variables : New developments *An article presented at the International Biometric Society-Kampala Workshop Kampala Uganda Dec.2001.*

Smoothing of Rainfall for three East African Countries (1961 - 1990)

Figure (1)

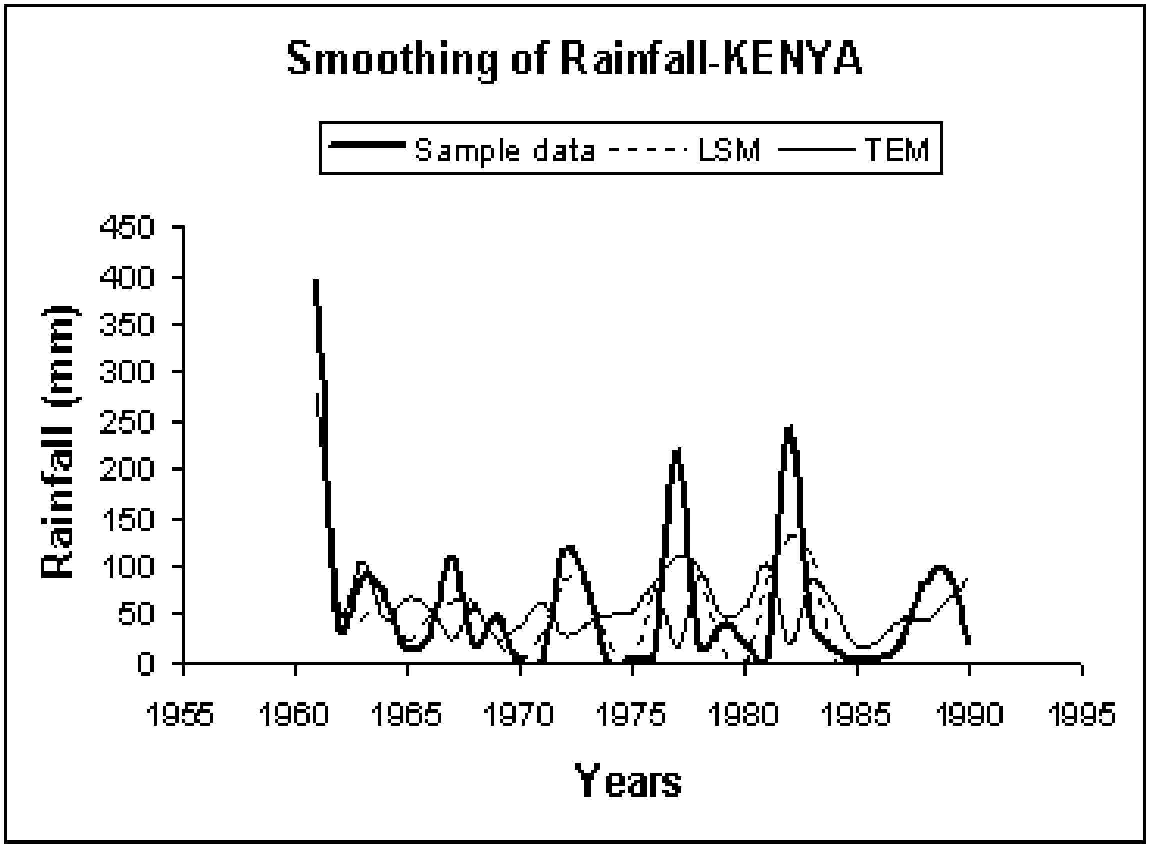
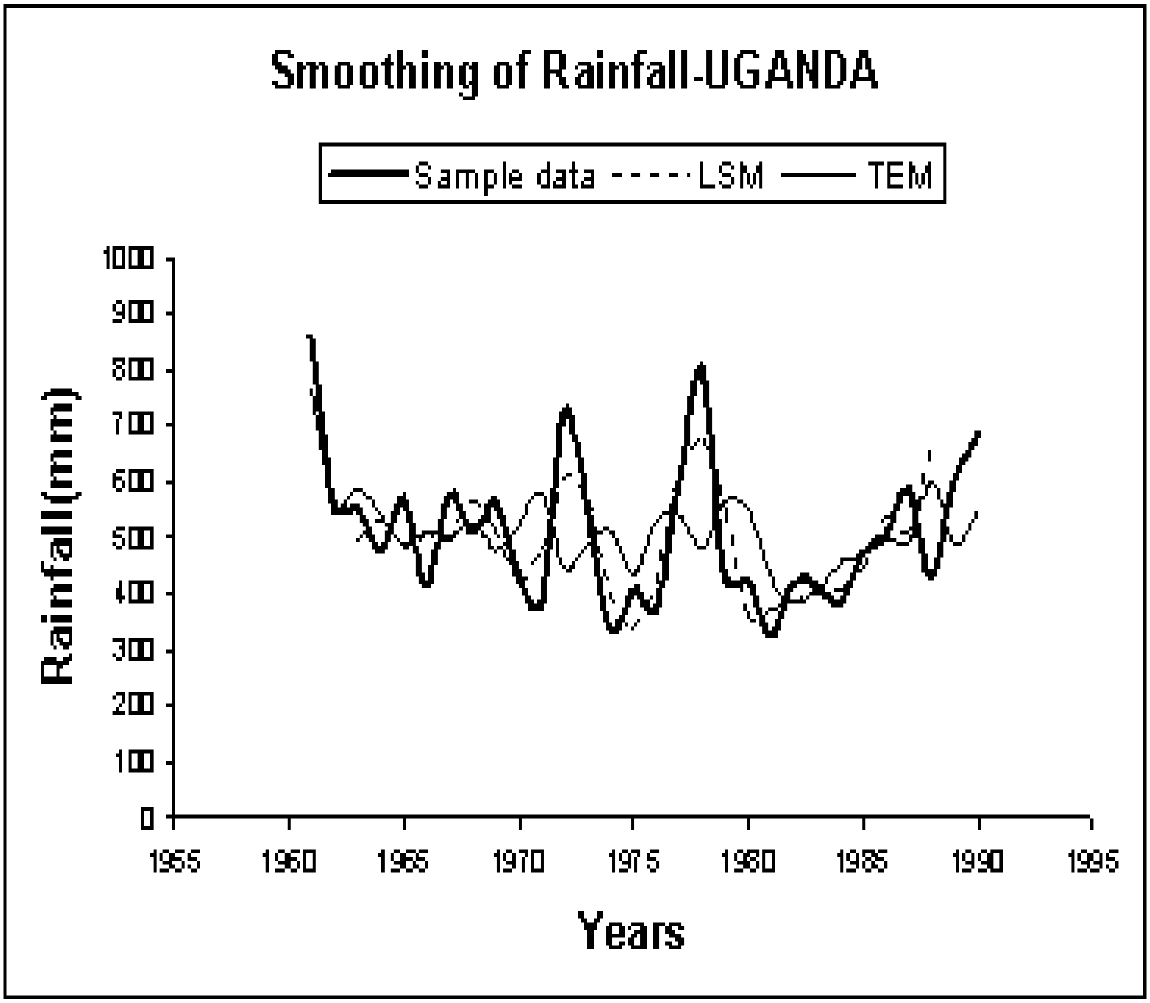
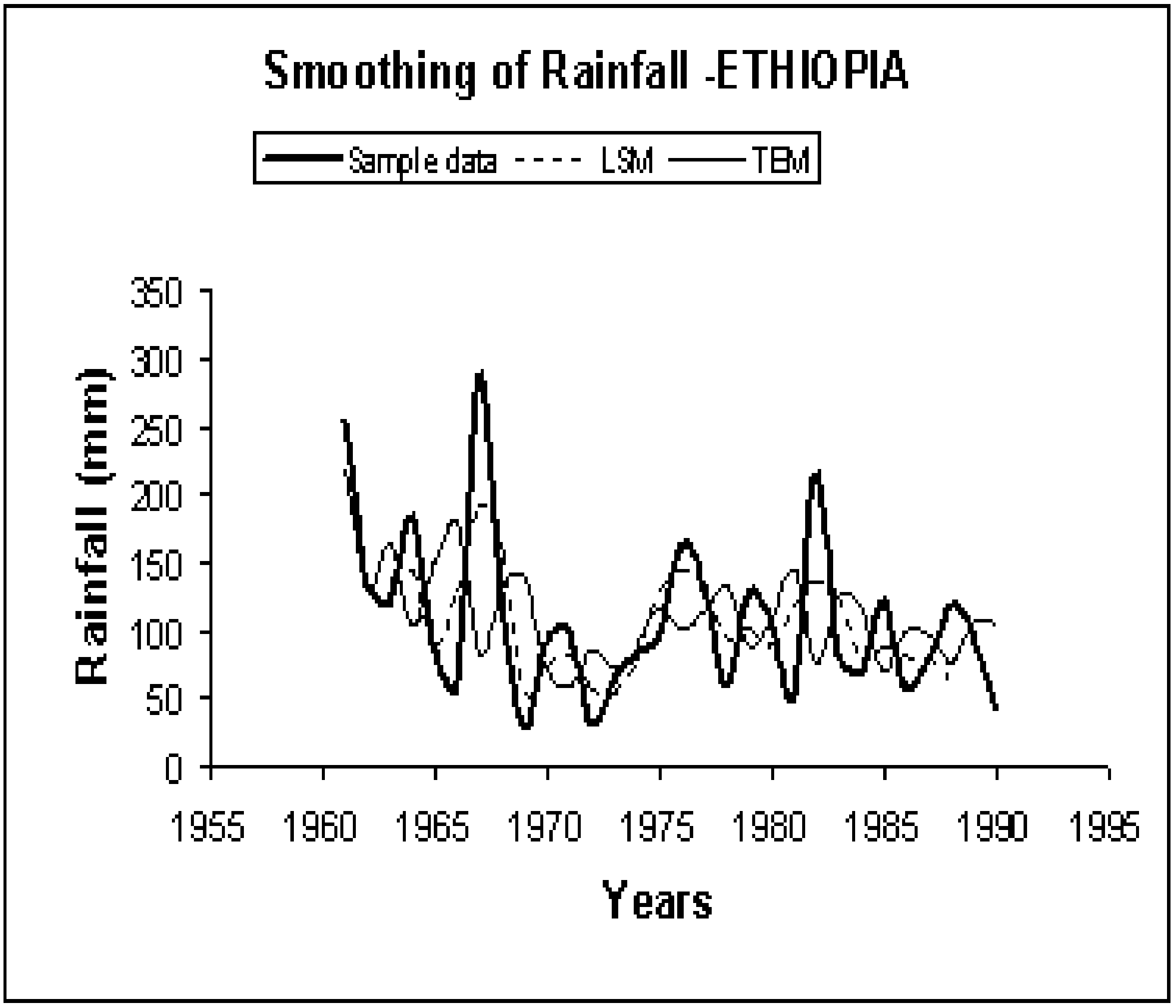


Figure (2)



Smoothing of Rainfall-UGANDA
Sample data
LSM
TEM
Rainfall(mm)
1000
900
800
700
600
500
400
300
200
100
0
1955
1960
1965
1970
1975
1980
1985
1990
1995
Years

Figure(3)


Smoothing of Rainfall -ETHIOPIA
Sample data
LSM
TBM
Rainfall (mm)
350
300
250
200
150
100
50
0
1955
1960
1965
1970
1975
1980
1985
1990
1995
Years