Information Matrices of Irregular Factorial Designs

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Abstract: A new method is presented for calculating the elements of T′T, of an array T in two symbols. The concept of marginal index is introduced. It is proved that the elements of T′T can be written in terms of marginal indices. A theorem is presented for the case of Partially Balanced array and its generalization for PB1 and Extended PB1 arrays. This method is useful in calculating the information matrix or parts of the information matrix of irregular factorial designs with two level factors.

Keywords: Design of Experiments, Factorial Design, Partially Balanced Array, PB1 Array, EPB1 Array, Information matrix.

1. Introduction

Bose and Srivastava (1964) obtained T′T of a Partially Balanced array T (Chakravarti (1956)) using the multidimensional partially balanced (MDPB) association scheme, a generalization of the usual association scheme. An N x m (0, 1) matrix T is said to be a Partially Balanced array (B-array) of strength t, size N, m constraints, 2 levels and index set { μ_0, μ_1,..., μ_t}, denoted as B(N, m, 2, t), if every subarray $T_{i1\ i2 \dots i\ t}$ of T is such that every (0, 1) vector with weight i (i= 0,..., t) occurs exactly μ_i times as a row of $T_{i1\ i2 \dots i\ t}$. Necessary and sufficient conditions for the existence of a B-array of strength t, m factors, 2 symbols and index set { μ_i | i = 0,..., t } are given by Srivastava (1972) for m≤ t+ 2, and Shirakura (1977) for m= t+ 3. Orthogonal arrays are B-arrays with $\mu_i = \mu$, $\forall$ i= 0,..., t. Srivastava (1970) expressed the elements of T′T (information matrix) of a B-array of strength t= 4 (resolution V design) in terms of its index set. Shirakura and Kuwada (1976) gave the information matrix of B-arrays that correspond to higher odd resolution designs, using the algebraic structure of the triangular multidimensional partially balanced (TMDPB) association scheme; their work includes that of Srivastava and Chopra (1971). Shirakura (1976b) gave the information matrix of B-arrays that correspond to designs of even resolution.

Kuwada (1988, 1988b) obtained the information matrix of PB1 designs of resolution V and VII using the algebraic structure of extended TMDPB (ETMDPB) association scheme. A (0, 1) matrix [$T^{(1)}$; $T^{(2)}$] of size N×(m_1+ m_2), in which $T^{(k)}$ (k= 1, 2) are of size N× m_k, is called a PB1 array of strength ($t_1+ t_2$), size N, $m_1+ m_2$ constraints, 2 levels and index set { $\mu(i_1, i_2) \mid 0 \le i_k \le t_k$, k= 1, 2 }, written PB1(N, $m_1+ m_2$, 2, $t_1+ t_2$, { $\mu(i_1, i_2)$ }), if for fixed values of t_k ($\le m_k$), every submatrix [$T_0^{(1)}$; $T_0^{(2)}$] of size N× ($t_1 + t_2$) is such that every (0, 1) vector with weight i_k in $T_0^{(k)}$ occurs exactly $\mu(i_1, i_2)$ times as a row of [$T_0^{(1)}$; $T_0^{(2)}$], where $T_0^{(k)}$ (k= 1, 2) are of size N× t_k and are consist of t_k columns of $T^{(k)}$ and the weight of a (0, 1) vector is the number of ones in the vector. (Kuwada and Kuriki (1986)). Necessary and sufficient conditions for the existence of PB1 arrays have been obtained by Kuwada and Kuriki (1986) using an argument similar to Srivastava (1972).

A generalization of PB1 array, the Extended PB1 (EPB1) array was given by Katsaounis (1999). A (0, 1) matrix [$T^{(1)}$;... ; $T^{(q)}$] of size N× (m_1 +...+ m_q), in which $T^{(k)}$ (k= 1, 2,..., q) are of size N× m_k, is called an Extended Partially Balanced Array of strength (t_1+...+ t_q), size N, m_1+...+ m_q constraints (factors), 2 levels and index set { $\mu(i_1,..., i_q) \mid 0 \le i_k \le t_k$, k= 1, 2,..., q }, written EPB1(N, m_1+...+m_q, 2, t_1+...+ t_q, { $\mu(i_1,..., i_q)$ }), if for fixed values of t_k ($\le m_k$), every submatrix [$T_0^{(1)}$;...; $T_0^{(q)}$] of size N× (t_1+...+ t_q) is such that every (0, 1) vector with weight i_k in $T_0^{(k)}$ occurs exactly $\mu(i_1,..., i_q)$ times as a row of [$T_0^{(1)}$;...; $T_0^{(q)}$] where $T_0^{(k)}$ (k= 1,..., q) are of size Nx t_k and consist of t_k columns of $T^{(k)}$.

In this paper, the concept of marginal index is introduced (Definition 2.1). Let $(x_{u_1})^{r_1} \ldots (x_{u_v})^{r_p}$ be the product of columns $x_{u_1}, \ldots, x_{u_v}$ of T, { $u_1, \ldots, u_v$ } $\subseteq$ { 1,…, m }, each raised to $r_1^{st}, \ldots, r_p^{th}$ power respectively; the quantity $\sum (x_{u_1})^{r_1} \ldots (x_{u_v})^{r_p}$ is the usual sum of products of order $r_1+\ldots+ r_p$ in design of experiments. Here, $\sum x_{u_1} \ldots x_{u_v}$ is considered, since a vector with 0's and 1's raised to a positive power is equal to itself and without loss of generality the two symbols in T can be taken to be 0 or 1. For a (0, 1) array T, marginal indices can be used to calculate the sum of the product of any columns of T, and thus specific elements or parts of T'T (Theorem 2.1). It is shown that, if T is B-array, or PB1 or EPB1 array, it can also be written as EPB1(N, m_1+...+m_q, 2, s_1+...+s_q, { $\mu(j_1,..., j_q)$ }) array with s_k= 2, $j_k \in$ { 0, 1, 2 } if $m_k \ge 2$ and

s_k= 1, $j_k \in$ { 0, 1} if m_k= 1, k= 1,…, q (see Theorem 2.2). If $m_k \geq 2$, s_k can be taken to be 1 or 2, and in general representation of $T^{(k)}$ as being of strength s_k is not unique, and thus marginal indices can be derived in different ways, depending on the choice of s_k. It is proved that such representations of T allow computation of any element of T′T, in terms of marginal indices (Propositions 2.1, 2.3, and 2.4).

2. Results

The array consisting of all $\binom{m}{i}$ (row) vectors of weight i, denoted by T_i^m, is called unit Simple array (Katsaounis (2002)). It is a B-(N, m, 2, m) array with index set {μ_w / $\exists$ i $\in$ { 0,…, m } such that μ_w= 1 if w= i, and 0 otherwise, w= 0,…, m}. A generalization of unit Simple array is the Compound array (Katsaounis (2002)), denoted by $T_{i_1,\ldots,i_q}^{m_1,\ldots,m_q}$ = [$T_{i_1}^{m_1}$;…; $T_{i_q}^{m_q}$]. A Compound array consists of the Cartesian product of $\binom{m_1}{i_1}$ distinct row vectors with weight i_1, $\binom{m_2}{i_2}$ ones with weight i_2, …, and $\binom{m_q}{i_q}$ ones with weight i_q, where $0 \leq i_k \leq m_k$. (2.1)

Definition 2.1 *Let T= [$T^{(1)}$;...; $T^{(q)}$] be an EPB1(N, m_1 +...+ m_q, 2, t_1 +...+ t_q, { μ(i_1,..., i_q) }), array with $1 \leq t_k \leq m_k$ and q ≥ 2. The index set of EPB1(N, m_{k_1} +...+ m_{k_p} , 2, t_{k_1} + … + t_{k_p} , { μ(i_{k1},..., i_{kp}) }) subarray T*= [$T^{(k_1)}$;...; $T^{(k_p)}$] , with { k_1,…, k_p} $\subseteq$ { 1,…, q }, is said to be marginal index set of order p.*

In the following $\mu_{i_1,\ldots,i_q}^{t_1,\ldots,t_q}$ is used instead of μ(i_1,…, ,i_q) to emphasize t_1,…, t_q.

Theorem 2.1 *If T*= [$T^{(k_1)}$;...; $T^{(k_p)}$], { k_1,…, k_p }$\subseteq$ { 1,…, q } is an EPB1(N, m_{k_1} +...+ m_{k_p} , 2, t_{k_1} + … + t_{k_p} , { $\mu^{t_{k1},\dots,t_{kp}}_{i_{k1},\dots,i_{kp}}$ }) subarray of EPB1(N, m_1+...+ m_q, 2, t_1+...+ t_q, { $\mu^{t_1,\dots,t_{k1},\dots,t_{kp},\dots,t_q}_{i_1,\dots,i_{k1},\dots,i_{kp},\dots,i_q}$ }), array T= [$T^{(1)}$;...; $T^{(q)}$] with $1\le t_k \le m_k$, then its marginal index set of order p is:*

$$\{ \mu^{t_{k1},\dots,t_{kp}}_{i_{k1},\dots,i_{kp}} \ / \ \forall\ (i_{k_1},\dots, i_{k_p}):\ \mu^{t_{k1},\dots,t_{kp}}_{i_{k1},\dots,i_{kp}} = \sum_{i_k\in A-A^*} \delta\left(T^{t_1,\dots,t_{k1},\dots,t_{kp},\dots,t_q}_{i_1,\dots,i_{k1},\dots,i_{kp},\dots,i_q} \right) \prod_{i_k\in A-A^*, t_k\in B-B^*} \binom{t_k}{i_k} \mu^{t_1,\dots,t_{k1},\dots,t_{kp},\dots,t_q}_{i_1,\dots,i_{k1},\dots,i_{kp},\dots,i_q},$$

where $\delta\left(T^{t_1,\dots,t_{k1},\dots,t_{kp},\dots,t_q}_{i_1,\dots,i_{k1},\dots,i_{kp},\dots,i_q} \right)$= 1 if $T^{t_1,\dots,t_{k1},\dots,t_{kp},\dots,t_q}_{i_1,\dots,i_{k1},\dots,i_{kp},\dots,i_q}$ is a subarray of T and it is 0 otherwise,

A= (i_1,..., i_q), A= (i_{k_1} ,..., i_{k_p})$\subseteq$ A, B= { t_1,..., t_q} and B*= { t_{k_1} ,..., t_{k_p} }$\subseteq$ B.*

Proof Partition T (row wise) into all Compound arrays $T^{t_1,\dots,t_{k1},\dots,t_{kp},\dots,t_q}_{i_1,\dots,i_{k1},\dots,i_{kp},\dots,i_q}$ (Katsaounis (2002)).

Given (i_{k_1} ,..., i_{k_p}) $T^{t_{k1},\dots,t_{kp}}_{i_{k1},\dots,i_{kp}}$ occurs $\prod_{i_k\in A-A^*, t_k\in B-B^*} \binom{t_k}{i_k}$ times within $T^{t_1,\dots,t_{k1},\dots,t_{kp},\dots,t_q}_{i_1,\dots,i_{k1},\dots,i_{kp},\dots,i_q}$ (by the definition of Compound array, see (2.1)). By the definition of EPB1 array, each subarray [$T_0^{(1)}$;...; $T_0^{(q)}$], of size N$\times$ (t_1+...+ t_q) is such that, every (0, 1) vector with weight i_k in $T_0^{(k)}$ occurs exactly $\mu^{t_1,\dots,t_q}_{i_1,\dots,i_q}$ times as a row of [$T_0^{(1)}$;...; $T_0^{(q)}$], where $T_0^{(k)}$ (k=1, 2,..., q) are of size Nx t_k and consist of t_k columns of $T^{(k)}$, i_k= 0, …, t_k, 1≤ t_k ≤ m_k. Thus, $T^{t_1,\dots,t_q}_{i_1,\dots,i_q}$ occurs exactly $\mu^{t_1,\dots,t_q}_{i_1,\dots,i_q}$ times or equivalently $T^{t_1,\dots,t_{k1},\dots,t_{kp},\dots,t_q}_{i_1,\dots,i_{k1},\dots,i_{kp},\dots,i_q}$ occurs $\mu^{t_1,\dots,t_{k1},\dots,t_{kp},\dots,t_q}_{i_1,\dots,i_{k1},\dots,i_{kp},\dots,i_q}$ total times in T**.** By keeping (i_{k_1} ,..., i_{k_p}) fixed and summing over all possible (i_1,..., i_q) in *A- A** (i.e. those such that (i_{k_1} ,..., i_{k_p})$\subseteq$ (i_1,…, i_{k_1} ,..., i_{k_p} ,..., i_q), the total number of times that $T^{t_{k1},\dots,t_{kp}}_{i_{k1},\dots,i_{kp}}$ occurs in T is:

$\mu^{t_{k1},\dots,t_{kp}}_{i_{k1},\dots,i_{kp}} = \sum_{i_k\in A-A^*} \delta\left(T^{t_1,\dots,t_{k1},\dots,t_{kp},\dots,t_q}_{i_1,\dots,i_{k1},\dots,i_{kp},\dots,i_q} \right) \prod_{i_k\in A-A^*, t_k\in B-B^*} \binom{t_k}{i_k} \mu^{t_1,\dots,t_{k1},\dots,t_{kp},\dots,t_q}_{i_1,\dots,i_{k1},\dots,i_{kp},\dots,i_q}$. The marginal index set of order p is obtained by considering all possible (i_{k_1} ,..., i_{k_p}). □

Theorem 2.2 *A B-(N, m, 2, t) array T with index set { μ_0, …, μ_t}, 1≤ t ≤ m can be written as EPB1(N, m_1 +...+ m_q, 2, s_1 +...+ s_q, { $\mu^{s_1,\ldots,s_q}_{j_1,\ldots,j_q}$ }), with m= m_1+...+ m_q, s_k=1 or 2, $j_k \in$ { 0, 1, 2 } if s_k= 2 and $j_k \in$ { 0, 1 } if s_k= 1.*

Proof Let m_1,..., m_q be positive integers, such that m_1 +...+ m_q= m . T can be partitioned (row wise) into unit Simple arrays T^m_i (Katsaounis (2001)). Each T^m_i can be written as the Cartesian product of $\binom{m_1}{i_1}$ distinct row vectors with weight i_1, $\binom{m_2}{i_2}$ ones with weight i_2, …, and $\binom{m_q}{i_q}$ ones with weight i_q , where 0≤ i_k ≤m_k and i= i_1 +...+ i_q, thus T can always be written as an EPB1 (N, m_1+...+m_q, 2, m_1 +...+ m_q, { $\mu^{m_1,\ldots,m_q}_{i_1,\ldots,i_q}$ }) array. Also, according to Theorem 2.1 Katsaounis (2001), T^m_i can be written as B-(N, m, 2, 2) array with index set { μ^2_0 , μ^2_1 , μ^2_2 }. It can be easily shown that T^m_i can also be written as B-(N, m, 2, 1) array with index set { μ^1_0, μ^1_1 }, where μ^1_0= μ^2_0 + μ^2_1 and μ^1_1= μ^2_1 + μ^2_2 . Thus, by choosing the same m_k, k= 1,…, q, for all T^m_i , T can be written as EPB1(N, m_1+...+ m_q, 2, s_1 +...+ s_q, { $\mu^{s_1,\ldots,s_q}_{j_1,\ldots,j_q}$ }), with $j_k \in$ { 0, 1, 2 } if s_k= 2 and $j_k \in$ { 0, 1 } if s_k= 1.□

Proposition 2.1 *If T is B-(N, m, 2, t) array T= [x_1,…, x_m] with index set { μ_0, …, μ_t }, 1≤ t ≤ m and u_1,…, $u_m \in$ { 1,…, m }, and n_{i_ℓ} is the number of columns in the product $x_{u_1} \ldots x_{u_m}$ corresponding to $T^{s_{i\ell}}_{j_{il}}$, ℓ =1,…,p , then $\sum x_{u_1} \ldots x_{u_m} = \sum_{C^*} \mu^{s_{i1},\ldots,s_{ip}}_{j_{i1},\ldots,j_{ip}}$, with*

$$\mu^{s_{i1},\ldots,s_{ip}}_{j_{i1},\ldots,j_{ip}} = \sum_{j_k \in C-C^*} \delta\,(T^{s_1,\ldots,s_{i1},\ldots,s_{ip},\ldots,s_q}_{j_1,\ldots,j_{i1},\ldots,j_{ip},\ldots,j_q}) \prod_{j_k \in C-C^*, s_k \in D-D^*} \binom{s_k}{j_k} \mu^{s_1,\ldots,s_{i1},\ldots,s_{ip},\ldots,s_q}_{j_1,\ldots,j_{i1},\ldots,j_{ip},\ldots,j_q},$$

where s_{i_1} ,…,s_{i_p} $\in$ { 1, 2 }, j_{i_1} ,…,j_{i_p} $\in$ { 1, 2 }, j_i ≤ s_i, i= 1,…, q, n_{i_ℓ} ≤ j_{i_ℓ} ≤ s_{i_ℓ} , C= (j_1,..., j_q), C= (j_{i_1} ,...,j_{i_p}) ⊆ C, D= { s_1,..., s_q} and D*= { s_{i_1} ,...,s_{i_p} } ⊆ D.*

Proof It follows from Theorem 2.1 and Theorem 2.2.□

Proposition 2.2 *If T= [$x_1,\ldots, x_m$] is B-(N, m, 2, t) array with index set { $\mu_0, \ldots, \mu_t$ },1≤ t ≤ m, then* $\sum x_{u_1} = \mu_1^2 + \mu_2^2$, $\sum x_{u_1} x_{u_2} = \mu_2^2$, $u_1, u_2 \in \{1,\ldots,m\}$.

Proof According to Proposition 2.1 Katsaounis (2001), T can be written as B-(N, m, 2, 2) array with index set { μ_0^2 , μ_1^2 , μ_2^2 }. Thus, $\sum x_{u_1} = \mu_1^2 + \mu_2^2$ i.e. the number of 1's in a column x_{u_1} of T and $\sum x_{u_1} x_{u_2} = \mu_2^2$, i.e. the number of 1's in the product of any columns x_{u_1} and x_{u_2} of T. □

Proposition 2.3 *Let T= [$T^{(1)}$; $T^{(2)}$]= [$x_1^{(1)}, \ldots, x_{m_1}^{(1)}$; $x_1^{(2)}, \ldots, x_{m_2}^{(2)}$], where $x_{m_k}^{(k)}$ are the columns of $T^{(k)}$, k= 1, 2. If { $x_{u_1}^{(1)}, \ldots, x_{u_{m1}}^{(1)}$ } ⊆ { $x_1^{(1)}, \ldots, x_{m_1}^{(1)}$ } and { $x_{u_1}^{(2)}, \ldots, x_{u_{m2}}^{(2)}$ } ⊆ { $x_1^{(2)}, \ldots, x_{m_2}^{(2)}$ } and $n_{i_\ell}^k$ is the number of columns in the product $x_{u_1}^{(1)} \ldots x_{u_{m1}}^{(1)} x_{u1}^{(2)} \ldots x_{um_2}^{(2)}$ corresponding to $T_{j_{i\ell}^k}^{s_{i\ell}^k}$, ℓ =1,…,p_k , then* $\sum x_{u_1}^{(1)} \ldots x_{u_{m1}}^{(1)} x_{u1}^{(2)} \ldots x_{um_2}^{(2)}$ *)=* $\sum_{C^{1*} \bigcup C^{2*}} \mu_{j_{i1}^1,\ldots j_{ip1}^1, j_{i1}^2,\ldots,j_{ip2}^2}^{s_{i1}^1,\ldots s_{ip1}^1, s_{i1}^2,\ldots,s_{ip2}^2}$ *,with*

$$\mu_{j_{i1}^1,\ldots j_{ip1}^1, j_{i1}^2,\ldots,j_{ip2}^2}^{s_{i1}^1,\ldots s_{ip1}^1, s_{i1}^2,\ldots,s_{ip2}^2} = \sum_{j_\ell^k \in C^k - C^{k*}} \delta\left(T_{j_1^1,\ldots,j_{i1}^1,\ldots j_{ip1}^1,\ldots,j_{q1}^1, j_1^2,\ldots,j_{i1}^2,\ldots,j_{ip2}^2,\ldots,j_{q2}^2}^{s_1^1,\ldots,s_{i1}^1,\ldots s_{ip1}^1,\ldots,s_{q1}^1, s_1^2,\ldots,s_{i1}^2,\ldots,s_{ip2}^2,\ldots,s_{q2}^2} \right) \prod_{j_\ell^k \in C^k - C^{k*}, s_\ell^k \in D^k - D^{k*}, k=1,2} \binom{s_\ell^k}{j_\ell^k}$$

$$\mu_{j_1^1,\ldots,j_{i1}^1,\ldots j_{ip1}^1,\ldots,j_{q1}^1, j_1^2,\ldots,j_{i1}^2,\ldots,j_{ip2}^2,\ldots,j_{q2}^2}^{s_1^1,\ldots,s_{i1}^1,\ldots s_{ip1}^1,\ldots,s_{q1}^1, s_1^2,\ldots,s_{i1}^2,\ldots,s_{ip2}^2,\ldots,s_{q2}^2}$$

, where $s_1^k,\ldots, s_{q_k}^k \in \{1, 2\}$, $j_1^k,\ldots, j_{q_k}^k \in \{1, 2\}$, $j_\ell^k \le s_\ell^k$,

ℓ = 1,…,q_k , $n_{i_\ell}^k \le j_{i_\ell}^k \le s_{i_\ell}^k$ ℓ =1,…, p_k, $m_k = \sum_{\ell=1}^{q_k} s_\ell^k$, $C^k = (j_1^k,\ldots, j_{q_k}^k)$, $C^{k*} = (j_{i_1}^k,\ldots, j_{i_{pk}}^k) \subseteq C^k$,

$D^{k*} = \{ s_{i_1}^k,\ldots, s_{i_{pk}}^k \} \subseteq \{ s_1^k,\ldots, s_{q_k}^k \} = D^k$.

Proof $T^{(k)}$, k= 1, 2, is a B (N, m_k, 2, t_k) array (by definition of PB1 array; its index set can be calculated by Theorem 2.1 or Proposition 2.1 (Katsaounis (2001)). By Theorem 2.2 $T^{(k)}$ can be written as EPB1(N, m_1^k +...+ $m_{q_k}^k$, 2, s_1^k +...+ $s_{q_k}^k$, { $\mu_{j_1^k,\dots,j_{qk}^k}^{s_1^k,\dots,s_{qk}^k}$ }), with s_ℓ^k = 2 and $j_\ell^k \in$ { 0, 1, 2 }

or s_ℓ^k = 1and $j_\ell^k \in$ { 0, 1 }, with ℓ = 1,…, q_k. Result follows from Theorem 2.1.□

Example 2.1

$$T=\begin{array}{ccc|cc}
1 & 1 & 1 & 1 & 1 \\ \hline
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\ \hline
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\ \hline
1 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 \\ \hline
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{array}$$

Table a.

$T_{3,2}^{3,2}$,	$\mu_{3,2}^{3,2} = 1$
$T_{1,0}^{3,2}$	$\mu_{1,0}^{3,2} = 1$
$T_{0,1}^{3,2}$	$\mu_{0,1}^{3,2} = 1$
$T_{2,0}^{3,2}$	$\mu_{2,0}^{3,2} = 1$
$T_{0,1}^{3,2}$	$\mu_{0,1}^{3,2} = 1$

Table b.

$T_{2,2}^{2,2}$	$\mu_{2,2}^{2,2} = 1$
$T_{1,0}^{2,2}$	$\mu_{1,0}^{2,2} = 1$
$T_{0,0}^{2,2}$	$\mu_{0,0}^{2,2} = 1$
$T_{0,1}^{2,2}$	$\mu_{0,1}^{2,2} = 1$
$T_{2,0}^{2,2}$	$\mu_{2,0}^{2,2} = 1$
$T_{1,0}^{2,2}$	$\mu_{1,0}^{2,2} = 1$
$T_{0,1}^{2,2}$	$\mu_{0,1}^{2,2} = 1$

T is PB1 array with N= 11, m= 3+ 2, t= 3+ 2 and index set { $\mu_{3,2}^{3,2}$= 1, $\mu_{1,0}^{3,2}$= 1, $\mu_{0,1}^{3,2}$= 1, $\mu_{2,0}^{3,2}$= 1, $\mu_{0,1}^{3,2}$= 1, all other 0 } (Table a). It can be written as PB1 array with N= 11, m= 2+ 2, t= 2+ 2 and index set { $\mu_{2,2}^{2,2}$= 1, $\mu_{1,0}^{2,2}$ = 2, $\mu_{0,0}^{2,2}$= 1, $\mu_{0,1}^{2,2}$ = 2, $\mu_{2,0}^{2,2}$ = 1} (Table b). The first column in Tables a and b gives the Compound arrays in T for both cases.

Let x_1, x_2, x_3 in $T^{(1)}$ and x_4, x_5 in $T^{(2)}$. Then according to Proposition 2.3:

i∈ { 1, 2, 3 }: $\sum x_i = \binom{2}{2}\mu_{2,2}^{2,2} + \binom{2}{0}\mu_{1,0}^{2,2} + \binom{2}{0}\mu_{2,0}^{2,2} = 4$

i∈ { 4, 5 } : $\sum x_i = \binom{2}{2}\mu_{2,2}^{2,2} + \binom{2}{0}\mu_{0,1}^{2,2} = 3$

i, j ∈ { 1, 2, 3 }: $\sum x_i\ x_j = \binom{2}{0}\mu_{2,2}^{2,2} + \binom{2}{0}\mu_{2,0}^{2,2} = 2$

i, j ∈ { 4, 5 }: $\sum x_i\ x_j = \binom{2}{0}\mu_{2,2}^{2,2} = 1$

Similarly,

i∈ { 1, 2, 3 } and j∈ { 4, 5 } : $\sum x_i\ x_j = \mu_{2,2}^{2,2} + \mu_{1,1}^{2,2} + \mu_{1,2}^{2,2} + \mu_{2,1}^{2,2} = 1$

i∈ { 1, 2, 3 } and j, k ∈ { 4, 5 }: $\sum x_i\,(x_j\, x_k) = \mu_{1,2}^{2,2} + \mu_{2,2}^{2,2} = 1$

i, j∈ { 1, 2, 3 } and k ∈ { 4, 5 }: $\sum (x_i\, x_j)\, x_k = \mu_{2,1}^{2,2} + \mu_{2,2}^{2,2} = 1$

i, j∈ { 1, 2, 3 } and k, ℓ ∈ { 4, 5 }: $\sum (x_i\, x_j)\,(x_k\ x_\ell) = \mu_{2,2}^{2,2} = 1$

and $\sum x_1 x_2 x_3 = \mu^{2,1}_{2,1} = \mu^{3}_{3} = 1$ (*)

(*) In general, calculations can be simplified using the fact that, if t= m, then $T^{t_1,\dots,t_q}_{i_1,\dots,i_q} = T^{t}_{i}$, $\mu^{t_1,\dots,t_q}_{i_1,\dots,i_q} = \mu^{t}_{i}$, where $t= t_1+\dots+t_q$, $i= i_1+\dots+i_q$. (this is a generalization of Remark (ii) Kuwada (1988), see also Katsaounis (2002)).

T'T=

	x_1	x_2	x_3	x_4	x_5	x_1x_2	x_1x_3	x_2x_3	x_4x_5
x_1	4	2	2	1	1	2	2	1	1
x_2	2	4	2	1	1	2	1	2	1
x_3	2	2	4	1	1	1	2	2	1
x_4	1	1	1	3	1	1	1	1	1
x_5	1	1	1	1	3	1	1	1	1
x_1x_2	2	2	1	1	1	2	1	1	1
x_1x_3	2	1	2	1	1	1	2	1	1
x_2x_3	1	2	2	1	1	1	1	2	1
x_4x_5	1	1	1	1	1	1	1	1	1

. □

Proposition 2.4 *Let* $T= [T^{(1)};\dots;T^{(q)}] = [\, x^{(1)}_{1},\dots, x^{(1)}_{m_1} ;\dots;\, x^{(q)}_{1},\dots, x^{(q)}_{m_q}\,]$*, where* $x^{(k)}_{u_{mk}}$ *are the columns of* $T^{(k)}$*, k= 1,…, q. If* $\{ x^{(k_v)}_{u_1} \dots x^{(k_v)}_{u_{m_{kv}}} \} \subseteq \{ x^{(k_v)}_{1} \dots x^{(k_v)}_{m_{kv}} \}$*,* $\{ k_1,\dots,k_v\} \subseteq \{ 1,\dots,q \}$*, v= 1,…,z, z= 1,…,q,* $n^{k_v}_{i_\ell}$ *is the number of columns in the product* $x^{(k_1)}_{u_1} \dots x^{(k_1)}_{u_{mk1}} \dots x^{(k_z)}_{u_1} \dots x^{(k_z)}_{u_{mkz}}$ *corresponding to* $T^{s^{kv}_{i\ell}}_{j^{kv}_{i\ell}}$*,* $\ell = 1,\dots,p_v$*, then*

$$\sum (x^{(k_1)}_{u_1} \dots x^{(k_1)}_{u_{mk1}}) \dots (x^{(k_z)}_{u_1} \dots x^{(k_z)}_{u_{mkz}}) = \sum_{\bigcup_{v=1}^{z} C^{kv*}} \mu^{s^{k1}_{i1},\dots s^{k1}_{ip1},\dots,s^{kz}_{i1},\dots,s^{kz}_{ipz}}_{j^{k1}_{i1},\dots j^{k1}_{ip1},\dots,j^{kz}_{i1},\dots,j^{kz}_{ipz}} ,$$

with

$$\mu^{s^{k1}_{i1},\dots s^{k1}_{ip1},\dots,s^{kz}_{i1},\dots,s^{kz}_{ipz}}_{j^{k1}_{i1},\dots j^{k1}_{ip1},\dots,j^{kz}_{i1},\dots,j^{kz}_{ipz}} = \sum_{\bigcup_{v=1}^{z} C^{kv}} \delta\left(T^{s^{k1}_{1},\dots,s^{k1}_{i1},\dots s^{k1}_{ip1},\dots,s^{k1}_{q1},\dots,s^{kz}_{1},\dots,s^{kz}_{i1},\dots,s^{kz}_{ipz},\dots,s^{kz}_{qz}}_{j^{k1}_{1},\dots,j^{k1}_{i1},\dots j^{k1}_{ip1},\dots,j^{k1}_{q1},\dots,j^{kz}_{1},\dots,j^{kz}_{i1},\dots,j^{kz}_{ipz},\dots,j^{kz}_{qz}} \right)$$

$$\prod_{j^{kv}_{\ell} \in C^{kv} - C^{kv*},\, s^{kv}_{\ell} \in D^{kv} - D^{kv*},\, k_v = k_1,\dots,k_z} \binom{s^{kv}_{\ell}}{j^{kv}_{\ell}} \mu^{s^{k1}_{1},\dots,s^{k1}_{i1},\dots s^{k1}_{ip1},\dots,s^{k1}_{q1},\dots,s^{kz}_{1},\dots,s^{kz}_{i1},\dots,s^{kz}_{ipz},\dots,s^{kz}_{qz}}_{j^{k1}_{1},\dots,j^{k1}_{i1},\dots j^{k1}_{ip1},\dots,j^{k1}_{q1},\dots,j^{kz}_{1},\dots,j^{kz}_{i1},\dots,j^{kz}_{ipz},\dots,j^{kz}_{qz}}$$

, where $s^{k_v}_{1},\dots, s^{k_v}_{q_v} \in \{ 1, 2 \}$*,* $\{ j^{k_v}_{1},\dots, j^{k_v}_{q_v} \} \in \{ 1, 2 \}$*,* $j^{k_v}_{\ell} \le s^{k_v}_{\ell}$*,* $\ell = 1,\dots,q_v$*,* $n^{k_v}_{i_\ell} \le j^{k_v}_{i_\ell} \le s^{k_v}_{i_\ell}$ $\ell = 1,\dots, p_v$*,* $\forall$ *v= 1,…,z :*

$m_k = \sum_{\ell=1}^{q_v} s^{k_v}_{\ell}$ *and* $C^{k_v} = (j^{k_v}_{1},\dots, j^{k_v}_{q_v})$*,* $C^{k^*_v} = (j^{k_v}_{i_1},\dots, j^{k_v}_{i_{pv}}) \subseteq C^{k_v}$*,* $D^{k^*_v} = \{ s^{k_v}_{i_1},\dots, s^{k_v}_{i_{pv}} \} \subseteq \{ s^{k_v}_{1},\dots, s^{k_v}_{q_v} \} = D^{k_v}$ *.*

Proof $T^{(k)}$, k= 1, ..,q is a B (N, m_k, 2, t_k) array (by definition of EPB1 array; its index set can be calculated by Theorem 2.1 or Proposition 2.1 (Katsaounis (2001)). By Theorem 2.2 $T^{(k)}$ can be written as EPB1(N, m_1^k +...+ $m_{q_k}^k$, 2, s_1^k +...+ $s_{q_k}^k$, { $\mu_{j_1^k,\ldots,j_{qk}^k}^{s_1^k,\ldots,s_{qk}^k}$ }), with s_ℓ^k = 2 and $j_\ell^k \in$ { 0, 1, 2 } or s_ℓ^k = 1and $j_\ell^k \in$ { 0, 1}, with ℓ = 1,…, q_k. Result follows from Theorem 2.1.□

Remark 2.1 In principle, if one can assign the columns of a B-(N, m, 2, t) array T into 'groups' of strength t and size t and if the number of columns from T is a multiple of t, $\sum x_{u_1} \ldots x_{u_v}$ = $\mu_{t,\ldots,t}^{t,\ldots,t}$, while if it is b mod t, then $\sum x_{u_1} \ldots x_{u_v} = \binom{t-1}{b-1} \mu_{t,\ldots,t,b}^{t,\ldots,t,t} + \ldots + \mu_{t,\ldots,t}^{t,\ldots,t}$. Thus if t= 2 and the number of columns is even, $\sum x_{u_1} \ldots x_{u_v} = \mu_{2,\ldots,2}^{2,\ldots,2}$, while if the number of columns is odd $\sum x_{u_1} \ldots x_{u_v} = \mu_{2,\ldots,2,1}^{2,\ldots,2,2} + \mu_{2,\ldots,2}^{2,\ldots,2}$. On the other hand, one can always assign the columns of T into 'groups' of strength 2 or 1 (and size 2 or 1) respectively (by Proposition 2.1 (Katsaounis (2001)). In the case of PB1 and EPB1 array, an obvious generalization holds for each $T^{(k)}$, since it is B-(N, m_k, 2, t_k) array by definition.

3. Conclusion

Marginal indices provide a way to calculate sums of products of columns of an array in T in two symbols, thus elements of T′T or parts of T′T. In general, T can be written as an EPB1 (N, m_1 +...+ m_q, 2, t_1 +...+ t_q, { μ(i_1,..., i_q) }) array, for some t_k, 2≤ t_k ≤ t'_k (t'_k full strength, Srivastava (1972)), for all k= 1,…, q, so that there is no unique representation of T as an EPB1 array. As a result, marginal indices and consequently sums of products of columns of T can be computed in different ways. In this paper, we illustrate how representation of T as an EPB1 (N, m_1 +...+ m_q, 2, s_1 +...+ s_q, { μ (j_1,..., j_q) }) array, with s_k= 1 or 2, k= 1,…, q, can be used to calculate efficiently marginal indices and thus T′T. Calculation of T′T can be simplified further by representing T as an EPB1(N, m_1 +...+ m_q, 2, s_1 +...+ s_q, { μ (j_1,..., j_q) }) array, with m_k = s_k= 1 or 2, k= 1,…, q.

In design of experiments, the analysis of a factorial design (solving the normal equations for a linear model) is based on evaluating the information matrix (T′T) and its inverse. Using marginal indices, one can readily examine any of the non diagonal elements of the information matrix, to determine the factors that result in a design with the desired properties for all factors or for a group (or groups) of factors.

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