# On Partially Balanced arrays

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**Abstract**:

Partially Balanced arrays with N runs, m factors, two symbols, strength t greater than two and a given index are discussed as being Partially Balanced arrays with N runs, m factors, two symbols and of strength equal to two. Formulas for calculating the corresponding index, given any number of factors m, are presented. This method is useful in showing easily non-existence of a Partially Balanced array with N runs, m factors, two symbols, strength t greater than two and a given index. Construction of irregular factorial designs with two levels that belong to the class of Partially Balanced arrays is also discussed.

*Keywords*: Irregular Factorial Design, Partially Balanced array, Simple array.

## 1. **Introduction**

An important subclass of irregular factorial designs, the balanced design (B-array) was first introduced by Chakravarti (1956), who gave them the name "partially balanced array", as a generalization of the orthogonal design. A B-array of strength t is defined as follows: An N x m (0, 1) matrix T is said to be a B-array of strength t, size N, m constraints (factors), 2 symbols (levels) and index set $\{\mu_0, \mu_1, ..., \mu_t\}$, written as B- (N, m, 2, t) array, if every subarray $T_{i1\ i2...it}$ of T is such that every (0, 1) vector with weight i= 0, 1,..., t occurs exactly $\mu_i$ times as a row of $T_{i1\ i2...it}$. The weight of a vector is the number of ones in that vector.

The information matrix T′ T of a B- array T is fully determined by its index set (Srivastava (1970), Yamamoto, Shirakura and Kuwada (1975)). An association between a B- array of strength t= 2 *l* (*l* is a positive integer) and a Balanced fractional $2^m$ factorial design of resolution 2*l* + 1 is given by Srivastava (1970) for *l* = 2, and by Yamamoto, Shirakura and Kuwada (1975) for $2l \leq m$. Necessary and sufficient conditions for the existence of a B- array are given by Srivastava (1972) for $m \leq t + 2$, and Shirakura (1977) for m= t + 3.

In general, a B- array of strength t is also of strength t', t' < t (Srivastava (1972)). For example, an N x m B- array of strength 4, can be also an N x m B- array of strength 2, although it might not be of strength 3. In this paper, we prove that any B- (N, m, 2, t), with $2 \leq t \leq m$ and index set $\{\mu_0, \ldots, \mu_t\}$ can always be written as a B- (N, m, 2, t'= 2) array with index set $\{\mu_0, \mu_1, \mu_2\}$ (Proposition 2.1). This result provides a computationally efficient method to examine if an N x m array in two symbols is a B- array. In section 3, we discuss a construction method of irregular factorial designs (with two levels) that belong to the class of B- arrays, as well as existence of an N x m B- array when an index set is given, based on this result.

## 2. **Results**

Let $T^{i:m}$ denote the array consisting of all $\binom{m}{i}$ (row) vectors of weight i, i= 0,…, m. Each $T^{i:m}$, i= 0,..., m, is a Simple array and also a B- array of strength t ((Shirakura (1977)). Here, we examine the case for t= 2 (Theorem 2.1). In the following, $\mu_w^{i:m}$ is used instead of $\mu_w$ (w= 0,…, t), when the value of m is emphasized.

**Theorem 2.1.** *$\forall$ i= 1,…, m and m≥ 2, $T^{i:m}$ can be written as a B- (N, m, 2, t= 2) array with index a) $\{\mu_0= 1, \mu_1= 0, \mu_2= 0\}$ if i= 0 b) $\{\mu_0= m-2, \mu_1= 1, \mu_2= 0\}$ if i= 1 and*

*c) $\mu_0=\binom{m}{i}-2\binom{m-2}{i-1}-\binom{m-2}{i-2}$, $\mu_1=\binom{m-2}{i-1}$ and $\mu_2=\binom{m-2}{i-2}$ if i≥ 2.*

**Proof.** Let m= 2. Trivially, $T^{0:2}=\begin{bmatrix}-1 & -1\end{bmatrix}$, $T^{1:2}=\begin{bmatrix}+1 & -1\\ -1 & +1\end{bmatrix}$, and $T^{2:2}=\begin{bmatrix}+1 & +1\end{bmatrix}$

(i= 0, 1, 2) are B- arrays of strength t= 2, and index sets $\{\mu_0= 1, \mu_1= 0, \mu_2= 0\}$, $\{\mu_0= 0, \mu_1= 1, \mu_2= 0\}$ and $\{\mu_0= 0, \mu_1= 0, \mu_2= 1\}$ respectively. (2.1)

Let m> 2. Let i= 0. $T^{0:m} = [\underbrace{-1 \quad \cdots \quad -1}_{m}]$ trivially can be written as B-(N= 1, m, 2, t= 2) array with index $\{\mu_0 = 1, \mu_1 = 0, \mu_2 = 0\}$. (2.2)

Let m> 2 and i ≥ 1. Let also 'a choose b' is equal to 0 if and only if 0≤ a< b or b< 0. Consider any two columns of $T^{i:m}$ and identify all weight 0, 1 and 2 vectors between them, so that $T^{i:m}$ is partitioned as follows:

$T^{i:m} = \begin{bmatrix} P(0,2) & P(i,m-2) \\ P(1,2) & P(i-1,m-2) \\ P(2,2) & P(i-2,m-2) \end{bmatrix}$. Thus, P(i-2, m-2) is the part corresponding to all weight 2 vectors, between the two selected columns, and is consisting of $\binom{m-2}{i-2}$ row vectors of weight i-2. Therefore, ∀ m> 2 $\mu_2 = \binom{m-2}{i-2}$. In a similar way, P(i-1, m-2) is the part corresponding to all weight 1 vectors and is consisting of $2\binom{m-2}{i-1}$ row vectors of weight i-1. Thus, ∀ m> 2 $\mu_1 = \binom{m-2}{i-1}$. Finally, P(i, m-2), the part corresponding to all weight 0 row vectors between any two columns of $T^{i:m}$, is consisting of $\mu_0 = \binom{m}{i} - 2\binom{m-2}{i-1} - \binom{m-2}{i-2}$ row vectors of weight i. This is derived using $N = \mu_0 + 2\mu_1 + \mu_2$, $N = \binom{m}{i}$. Thus, $T^{i:m}$ can be written as a B- (N = $\binom{m}{i}$, m, 2, t= 2) array with index

$\{\mu_0 = \binom{m}{i} - 2\binom{m-2}{i-1} - \binom{m-2}{i-2},\ \mu_1 = \binom{m-2}{i-1},\ \mu_2 = \binom{m-2}{i-2}\}$. (2.3)

In the special case, when i= 1, $T^{1:m}$ can be written as a B- (N= m, m, 2, t= 2) array with index $\{\mu_0 = \binom{m-2}{1}, \mu_1 = 1, \mu_2 = 0\}$ (2.4)

Theorem 2.1 holds by (2.1), (2.2), (2.3) and (2.4).

An array T consisting of $T^{i:m}$ arrays, $i \in \{0,\dots, m\}$, is by definition a Simple array and a B- (N, m, 2, t) array, with size $N= \sum_{i=0}^{t}\binom{t}{i}\mu_i$ (here t= m), strength t= m and index set $\{\mu^*_i \mid \mu^*_i = \sum_{i=0}^{m} \delta_i \mu_i$, $i= 0,\dots, t$, $t= m\}$, where $\delta_i= 1$ if $T^{i:m} \in T$, and 0 otherwise, since $\forall\, T^{i:m} \in T$ there are $\mu_i$ weight i vectors (see also Srivastava (1972) and Shirakura (1977)). Thus, by Theorem 2.1, an array T consisting of $T^{i:m}$ arrays, $i \in \{0,\dots, m\}$ can be written as a B- (N, m, 2, t= 2) array with index set

$$\{\mu_w \mid \mu_w = \sum_{i=0}^{m} \delta_i \mu^{i:m}_w,\ w= 0, 1, 2\}, \text{ where } \delta_i= 1 \text{ if } T^{i:m} \in T, \text{ and 0 otherwise} \qquad (2.5)$$

and $N= \sum_{w=0}^{2}\binom{2}{w} \mu^{i:m}_w$.

The main result is the following:

**Proposition 2.1.** *If T is a B- (N, m, 2, t) array, with $2 \le t \le m$ and index set $\{\mu^*_i$, i= 0,…, t\}, it can be written as a B- (N, m, 2, t= 2) array with index set $\{\mu_i$, i= 0, 1, 2\} as given in (2.5).*

**Proof.** Let T be a B- (N, m, 2, t) array, with $2 \le t \le m$ and index set $\{\mu^*_i$, i= 0,…, t\}. Consider m′= t (≤ m) factors. Then $T_{i1\, i2 \dots it}$ (as in the definition of B- array above) can be written as a B- (N, m′, 2, t= m′) array and index set $\{\mu^*_i \mid i= 0,\dots, t\}$. By Theorem 2.1, $T_{i1\, i2 \dots it}$ can be written as a B- (N, m′, 2, t= 2) array. By (2.5), it has size $N= \sum_{w=0}^{2}\binom{2}{w} \mu^{i:m}_w$, t= 2 and index set $\{\mu_w \mid \mu_w = \sum_{i=0}^{m} \delta_i \mu^{i:m}_w,\ w= 0, 1, 2\}$.

But this holds for any set of m′= t factors. Thus, by permuting the m factors and considering m′ at a time, it holds for the whole array T.

**Proposition 2.2**. *If T is an Orthogonal array (N, m, 2, t) with $2 \leq t \leq m$ then T can be written as a Orthogonal array (N, m, 2, t= 2) and index set {$\mu$, $\mu$, $\mu$}.*

**Proof.** This follows from Proposition 2.1 and the fact that an Orthogonal array is a special case of a B- array with $\mu^*_i = \mu$, $\forall i = 0, 1, \ldots, t$.

In the following Theorem 2.2, we give an efficient way for calculating $\{\mu_0, \mu_1, \mu_2\}$, for any m> 2, using only the index set of $T^{i:m}$'s, $i \in \{0, \ldots, [m/2]\}$, where [a] is the integer part of a real number a.

**Theorem 2.2.** *Given m, consider a B- (N, m, 2, t= 2) array T with index set $\{\mu_0, \mu_1, \mu_2\}$. Then:*

$$\mu_0 = \sum_{i=0}^{[m/2]} \delta_i \, \mu_0^{i::m} + \sum_{i=[m/2]+1}^{m} \delta_i \, \mu_0^{i::m} = \sum_{i=0}^{[m/2]} \delta_i \, \mu_0^{i::m} + \sum_{i=0}^{[m/2]} \delta_i \, \mu_2^{i::m} \qquad (i)$$

$$\mu_1 = \sum_{i=0}^{[m/2]} \delta_i \, \mu_1^{i::m} + \sum_{i=[m/2]+1}^{m} \delta_i \, \mu_1^{i::m} = 2 \sum_{i=0}^{[m/2]} \delta_i \, \mu_1^{i::m} \qquad (ii)$$

$$\mu_2 = \sum_{i=0}^{[m/2]} \delta_i \, \mu_2^{i::m} + \sum_{i=[m/2]+1}^{m} \delta_i \, \mu_2^{i::m} = \sum_{i=0}^{[m/2]} \delta_i \, \mu_2^{i::m} + \sum_{i=0}^{[m/2]} \delta_i \, \mu_0^{i::m} \qquad (iii)$$

**Proof.** Without loss of generality take the two symbols to be +1 and –1. By interchanging +1 and –1, the number of weight i vectors of $T^{i:m}$ is equal to the number of weight m-i vectors of $T^{m-i:m}$, thus only i= 0,…, [m/2] can be considered. Equation (i) holds since $\mu_0^{i::m} = \mu_2^{m-i::m}$ and therefore $\sum_{i=[m/2]+1}^{m} \delta_i \, \mu_0^{i::m} = \sum_{i=0}^{[m/2]} \delta_i \, \mu_2^{i::m}$, by rearranging the index in the summation. Similarly, $\sum_{i=0}^{[m/2]} \delta_i \mu_1^{i::m} + \sum_{i=[m/2]+1}^{m} \delta_i \mu_1^{i::m} = 2 \sum_{i=0}^{[m/2]} \delta_i \mu_1^{i::m}$, thus equation (ii) holds. Equation (iii) holds since $\mu_0^{i::m} = \mu_2^{m-i::m}$ thus $\sum_{i=[m/2]+1}^{m} \delta_i \mu_2^{i::m} = \sum_{i=0}^{[m/2]} \delta_i \mu_0^{i::m}$.

## 3. **Discussion and Conclusions**

Proposition 2.1 provides a computationally efficient way to examine whether or not an N x m array in two symbols is a B- array: it is sufficient to check the number of weight 0, 1 and 2 vectors between pairs of factors; if for any two pairs of factors, the number of weight 0, 1 and 2 (row) vectors is not the same, then the given array is not a B- (N, m, 2, 2) (by definition) and by Proposition 2.1 is not a B- (N, m, 2, t) array with $t \geq 2$ either (see also Srivastava (1972)). However, if the number of weight 0, 1 and 2 (row) vectors is the same for all pairs of factors, then (by definition) the given array is a B- ( N, m, 2, 2) with index set $\mu_i$, i= 0, 1, 2. However, if the given array is also a B- (N, m, 2, t) with $t \geq 2$, t has to be ‘adjusted’ in order to identify the ‘full’ strength of the array and the corresponding index set $\mu_i$, i= 0, …, t (see also Shirakura and Kuwada (1974)).

Proposition 2.1 provides also an efficient way to construct a B- (N, m, 2, t) array with a given index set $\mu'_i$, i= 0, …, t (and thus N given): Initially let m*= t (m*≤ m) and identify any $T^{i:m}$’s that yield a B- (N, m*, 2, t) array T with the given index set and corresponding (t= 2) index set $\mu_i$, i= 0, 1, 2. If there is no B- (N, m*, 2, t= 2) array with index set $\{\mu_0, \mu_1, \mu_2\}$, then there is no B- (N, m**, 2, t= 2) array with m**> m*, and index set $\{\mu_0, \mu_1, \mu_2\}$ (Srivastava (1972)). Thus, by Proposition 2.1, there is no B- (N, m**, 2, t) array with m**> m*, and $t \geq 2$ and index set $\{\mu_0, \dots, \mu_t\}$ also. If a B- (N, m**, 2, t) array exists, with m**> m*, it can be constructed by examining the number of weight 0, 1 and 2 vectors between pairs of candidates (factors), and identifying those that yield the desired index set $\mu_i$, i= 0, 1, 2. Additional criteria (e. g. optimality criteria) will limit the search for appropriate candidates.

B- arrays are generalizations of balanced incomplete block (BIB) designs, since the incidence matrix of a BIB design (with equal or unequal block sizes) is a B- array of strength 2 and 2 symbols ( Srivastava (1972)). Thus, existence conditions of BIB designs can be used to verify existence of B- arrays of strength 2.

## References


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