Methods for Estimating the Parameters of the Weibull Distribution

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Abstract

In this paper, we present some methods for estimating Weibull parameters, namely, shape parameter (b) and scale parameter (h). The Weibul distribution is an important distribution especially for reliability and maintainability analysis. The presented methods can be classified into two categories: graphical and analytical methods. Computational experiments on the presented methods are reported.

1. Graphical Methods

Usually, the graphical methods are used because of their simplicity and speed. However, they involve a great probability of error. Next we discuss two main graphical methods.

1.1 Weibull Probability Plotting

The Weibull distribution density function (Mann et al. (1974)) is given by:

$$f(x) = \frac{b}{h}\left(\frac{x-g}{h}\right)^{b-1} e^{-\left(\frac{x-g}{h}\right)^{b}} \quad , b > 0, h > 0,\ x \geq g \geq 0 \qquad (1)$$

The cumulative Weibull distribution function is given by:

$$F(x) = 1 - e^{-\left(\frac{x-g}{h}\right)^{b}} \qquad (2)$$

where; b is the shape parameter, h is the scale parameter, and g is the location parameter.

To come up with the relation between the CDF and the two parameters (b , h), we take the double logarithmic transformation of the CDF.

From (2) and letting g = 0, we have

$$1 - F(x) = e^{-(\frac{x}{h})^b}$$

$$\frac{1}{1 - F(x)} = e^{-(\frac{x}{h})^b}$$

$$\ln[\frac{1}{1 - F(x)}] = -(\frac{x}{h})^b$$

$$\ln \ln[\frac{1}{1 - F(x)}] = b \ln h - b \ln x$$

The last equation is an equation of a straight line. To plot $F(x)$ versus x, we apply the following procedure:

1. Rank failure times in ascending order.
2. Estimate $F(x_i)$ of the i[th] failure.
3. Plot $F(x_i)$ vs. x in the Weibull probability paper.

To estimate $F(x_i)$ in (2 and 3) above, we may use one of the following methods presented in Table 1 where n is number of data points.

Method	$F(x_i)$
Mean Rank	$\frac{i}{n+1}$
Median Rank	$\frac{i-0.3}{n+0.4}$
Symmetrical CDF	$\frac{i-0.5}{n}$

Table 1. Methods for estimating $F(x_i)$.

1.2 Hazard Plotting Technique

The hazard plotting technique is an estimation procedure for the Weibull parameters. This is done by plotting cumulative hazard function $H(x)$ against failure times on a hazard paper or a simple log-log paper.
The hazard function is given below:

$$h(x) = \frac{b}{h}(\frac{x}{h})^{b-1}$$

The cumulative hazard function is given below:

$$H(x) = \int h(x) = (\frac{x}{h})^b \tag{3}$$

We can transform (3) by taking the logarithm as follows

$$ln\ H(x) = b\{ln\ x - ln\ h\}$$

$$ln\ x = \frac{1}{b}\ ln\ H(x) + ln\ h \tag{4}$$

From (4), we can then plot $\ln H(x)$ versus $\ln x$ using the following procedure:

1. Rank the failure times in ascending order.
2. For each failure, calculate $\Delta H_i = \frac{1}{(n+1)-1}$
3. For each failure, calculate $H = \Delta H_1 + \Delta H_2 + \cdots + \Delta H_i$
4. Plot $ln\ H$ vs. $ln\ x$.
5. Fit a straight line.

Upon completing the plotting, the estimated parameters will be as follows:

$$b = \frac{x}{y} = \frac{1}{slope}$$

$$at\ H = 1, \quad h = x$$

2. Analytical Methods

Due to the high probability of error in using graphical methods, we prefer to use the analytical methods. This is motivated by the availability of high-speed computers. In the following, we discuss some of the analytical methods used in estimating Weibull parameters.

2.1 Maximum Likelihood Estimator (MLE)

The method of maximum likelihood (Harter and Moore (1965a), Harter and Moore (1965b), and Cohen (1965)) is a commonly used procedure because it has very desirable properties. Let x_1, x_2, ..., x_n be a random sample of size n drawn from a probability density function $f_X(x;q)$ where q is an unknown parameter. The likelihood function of this random sample is the joint density of the n random variables and is a function of the unknown parameter. Thus

$$L = \prod_{i=1}^{n} f_{X_i}(x_i, q) \tag{5}$$

is the likelihood function. The maximum likelihood estimator (MLE) of q, say $\hat{q}$, is the value of q that maximizes L or, equivalently, the logarithm of L. Often, but not always, the MLE of q is a solution of

$$\frac{d\ log\ L}{dq} = 0$$

where solutions that are not functions of the sample values x_1, x_2, ..., x_n are not admissible, nor are solutions which are not in the parameter space. Now, we are going to apply the MLE to estimate the Weibull parameters, namely the shape and the scale parameters. Consider the Weibull pdf given in (1), then likelihood function will be

$$L(x_1,\cdots,x_n;b,h) = \prod_{i=1}^{n}(\frac{b}{h})(\frac{x_i}{h})^{b-1}e^{-(\frac{x_i}{h})^b} \tag{6}$$

On taking the logarithms of (6), differentiating with respect to b and h in turn and equating to zero, we obtain the estimating equations

$$\frac{¶\ln L}{¶b} = \frac{n}{b} + \sum_{i=1}^{n}\ln x_i - \frac{1}{h}\sum_{i=1}^{n}x_i^b \ln x_i = 0 \tag{7}$$

$$\frac{¶\ \ln\ L}{¶h} = -\frac{n}{h} + \frac{1}{h^2}\sum_{i=1}^{n}x_i^b = 0 \tag{8}$$

On eliminating h between these two equations and simplifying, we have

$$\frac{\sum_{i=1}^{n}x_i^b \ln x_i}{\sum_{i=1}^{n}x_i^b} - \frac{1}{b} - \frac{1}{n}\sum_{i=1}^{n}\ln x_i = 0 \tag{9}$$

which may be solved to get the estimate of $\hat{\mu}_\kappa = b$. This can be accomplished by the use of standard iterative procedures (i.e., Newton-Raphson method). Once b is determined, h can be estimated using equation (8) as

$$h = \frac{\sum_{i=1}^{n}x_i^b}{n} \tag{10}$$

2.2 Method of Moments (MOM)

The method of moments is another technique commonly used in the field of parameter estimation. If the numbers x_1, x_2, ..., x_n represent a set of data, then an unbiased estimator for the kth origin moment is

$$\hat{m}_k = \frac{1}{n}\sum_{i=1}^{n}x_i^k \tag{11}$$

where; $\hat{m}_k$ stands for the estimate of m_k. In Weibull distribution, the kth moment readily follows from (1) as

$$m_k = \left(\frac{1}{h^b}\right)^{-\frac{k}{b}} \Gamma\left(1+\frac{k}{b}\right) \qquad (12)$$

where Γ signifies the gamma function

$$\Gamma(s) = \int_0^\infty x^{s-1} e^{-x} dx, (s > 0)$$

Then from (12), we can find the first and the second moment as follows

$$m_1 = \hat{m}_k = \left(\frac{1}{h}\right)^{\frac{1}{b}} \Gamma\left(1+\frac{1}{b}\right) \qquad (13)$$

$$m_2 = \hat{m}_k^2 + \hat{s}_k^2 = \left[\frac{1}{h}\right]^{\frac{2}{b}} \left\{ \Gamma\left(1+\frac{2}{b}\right) - \left[\Gamma\left(1+\frac{1}{b}\right)\right]^2 \right\} \qquad (14)$$

When we divide m_2 by the square of m_1, we get an expression which is a function of b only

$$\frac{\hat{s}_k^2}{\hat{m}_k^2} = \frac{\Gamma(1+\frac{2}{b}) - \Gamma^2(1+\frac{1}{b})}{\Gamma^2(1+\frac{1}{b})}$$

(15)

On taking the square roots of (15), we have the *coefficient of variation*

$$CV = \frac{\sqrt{\Gamma(1+\frac{2}{b}) - \Gamma^2(1+\frac{1}{b})}}{\Gamma(1+\frac{1}{b})} \qquad (16)$$

Now, we can form a table for various CV by using (16) for different b values. In order to estimate b and h, we need to calculate the *coefficient of variation* $(CV)_d$ of the data on hand. Having done this, we compare $(CV)_d$ with CV using the table. The corresponding b is the estimated one ($\hat{b}$). The scale parameter (h) can then be estimated using the following

$$\hat{h} = \left\{\bar{x} / \Gamma\left[(1/\hat{b}) + 1\right]\right\}^{\hat{b}}$$

where $\bar{x}$ is the mean of the data.

2.3 Least Squares Method (LSM)

The third estimation technique we shall discuss is known as the Least Squares Method. It is so commonly applied in engineering and mathematics problems that is often not thought of as an estimation problem. We assume that a linear relation between the two variables (see section 1). For the estimation of Weibull parameters, we use the method of least squares and we apply it to the results of section 1. Recall that

$$\ln \ln\left[\frac{1}{1-F(x)}\right] = b \ln x - b \ln h \qquad (17)$$

Equation (17) is a linear equation. Now, we can write

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} \ln\left\{\ln\left[\frac{1}{(1-\frac{i}{n+1})}\right]\right\} \tag{18}$$

$$\bar{Y} = \frac{1}{n}\sum_{i=1}^{n} \ln x_i \tag{19}$$

$$\hat{b} = \frac{\left\{n\cdot\sum_{i=1}^{n}(\ln x_i)\cdot(\ln\{\ln[\frac{1}{(1-\frac{i}{n+1})}]\})\right\} - \left\{\sum_{i=1}^{n}\ln(\ln[\frac{1}{(1-\frac{i}{n+1})}])\cdot\sum_{i=1}^{n}\ln x_i\right\}}{\left\{n\cdot\sum_{i=1}^{n}(\ln x_i)^2\right\} - \left\{\sum_{i=1}^{n}(\ln x_i)\right\}^2} \tag{20}$$

$$\hat{h} = e^{(\bar{Y}-\bar{x}/\hat{b})} \tag{21}$$

From equations (18)-(21), we can calculate the estimate of b and h .

3. Numerical Example

In order to illustrate and compare the methods described earlier, we have coded the three analytical methods MLE, MOM and LSM in BASIC Language and we have used a Pentium PC with 133MHz . To compare the three methods, we used the MSE (Mean Squared Error) test. MSE can be calculated as below

$$MSE = \sum_{i=1}^{n}\{\hat{F}(x_i) - F(x_i)\}^2 \tag{22}$$

where $\hat{F}(x_i) = 1 - e^{(-x_i/\hat{h})^{\hat{b}}}$ and $F(x_i) = \frac{i-0.3}{n+0.4}$

Example 1:

Consider the following example where x_i represents the i^{th} failure time.

Table 2. Data for example 1.

i	x_i
1	0.438
2	2.413
3	3.073
4	3.079
5	3.137
6	3.198
7	3.918
8	4.287
9	4.508
10	4.981

11	5.115
12	5.592
13	5.848
14	5.958
15	6.013

Table 3 below shows the complete results of the estimates of the shape and scale parameters and the mean squared error (MSE) for each method.

Table 3. Results of example 1.

Method	b	h	MSE
MLE	2.923	4.552	3.57×10^{-3}
MOM	2.941	4.587	3.17×10^{-3}
LSM	1.8515	4.756	8.4×10^{-3}

Since MOM has the minimum MSE, then $\hat{b}$ =2.941 and $\hat{h}$ =4.587.

In comparing our results with the well-known software called STATGRAPHICS, we found that STATGRAPHICS is using the MLE and our MLE estimates are exactly the same as the estimates of STATGRAPHICSs MLE. However, we have the advantage of having different estimates for different methods and the ability of choosing one of them.

4. Computational Results

The objective of our experiments is to compare the three methods namely, LSM, MLE and MOM. We have generated random samples with known parameters. For each sample, we have varied the size from 20 to 100. To be able to compare, we calculated the total deviation (TD) for each method as follows:

$$TD = \left|\frac{\hat{b} - b}{b}\right| + \left|\frac{\hat{h} - h}{h}\right| \quad (23)$$

where; b and h are the known parameters, and $\hat{b}$ and $\hat{h}$ are the estimated parameters by any method.

Table 4 shows the complete results. The last column of the table shows the best method which yields the minimum total deviation. Notice that the maximum *TD* is 0.55 for all methods. This means that at the worst case the estimated parameters are within $\pm 50\%$ of their actual values.

Table 5 shows a summary of the results of Table 4. As it is obviously seen, MOM is the best method. MOM achieves the best estimate 12 times out of 25 which is approximately 50% of the time. Its average deviation from the actual values is 17% with a standard deviation of 14% which is quite good. MOM achieves good results because it involves more calculations and require more computation time than LSM or MLE. However, for a sample of size 100, MOM takes only few seconds.

Table 4. Comparison between LSM, MLE and MOM

n	*b*	*h*	Size	LSM			MLE			MOM			Best
				b	*h*	TD	*b*	*h*	TD	*b*	*h*	TD	
1	1	10	20	1.34	7.08	0.63	1.22	7.3	0.5	1.1	7.09	0.4	MOM
			40	0.85	10.96	0.24	0.94	10.74	0.13	0.98	10.8	0.1	MOM
			60	0.84	12.17	0.37	0.926	11.87	0.26	0.94	11.9	0.25	MOM
			80	1.07	10.83	0.16	1.11	10.8	0.21	1.15	10.9	0.24	LSM
			100	0.87	11.18	0.24	0.93	10.97	0.16	0.93	10.8	0.15	MOM
2	2.3	145	20	2.36	180.3	0.27	2.7	178.4	0.41	2.7	178.1	0.4	LSM
			40	3.4	155.1	0.55	3.3	155.8	0.51	3.33	155.2	0.52	MLE
			60	2.19	132.5	0.13	2.38	131.8	0.13	2.35	131.6	0.11	MOM
			80	2.43	139.06	0.09	2.25	140.5	0.05	2.25	140.1	0.05	MLE
			100	2.44	156.9	0.14	2.62	155.6	0.21	2.6	155.6	0.2	MOM
3	2.9	357	20	2.91	358.7	0.01	3.35	354.9	0.16	3.33	354.1	0.15	LSM
			40	3.32	365.7	0.14	3.6	364.1	0.25	3.4	363.6	0.19	LSM
			60	2.63	373.8	0.13	2.95	370.5	0.05	2.9	370.4	0.04	MOM
			80	2.62	373.04	0.14	3.03	367.8	0.07	3.0	367.8	0.06	MOM
			100	2.7	367.6	0.1	3.02	363.7	0.06	3.0	367.4	0.05	MOM
4	3.5	1270	20	3.77	1276.4	0.08	4.4	1270.3	0.27	4.1	1264.6	0.19	LSM
			40	3.12	1200	0.16	3.57	1192.4	0.08	3.4	1189.6	0.09	MLE
			60	3.07	1250.7	0.13	3.3	1248.5	0.06	3.2	1245.3	0.1	MLE
			80	3.09	1304.5	0.14	3.8	1282.8	0.1	3.6	1284.2	0.04	MOM
			100	3.46	1230.9	0.04	3.52	1228.2	0.03	3.57	1223.5	0.05	MLE
5	1.9	872	20	1.5	888.8	0.22	1.83	867.7	0.03	1.9	870.6	0.01	MOM
			40	2.1	949.6	0.19	2.6	920.9	0.47	2.8	926.7	0.53	LSM
			60	2.3	957.3	0.32	2.25	962.3	0.3	2.25	959.9	0.28	MOM
			80	1.64	907.8	0.17	1.95	884.7	0.04	2.0	890.2	0.08	MLE
			100	1.91	827.3	0.06	1.96	824.8	0.08	1.98	824.8	0.1	LSM

Table 5. Summary of Results

Method	No. of times the method gives the best estimate	Average percentage of deviation from actual values (m)	Standard deviation of the deviations (s)
LSM	7	21 %	15 %
MLE	6	18 %	15 %
MOM	12	17 %	14 %

5. Method Selection

We have described two graphical methods and three analytical methods for estimating Weibull parameters b and h . Now, the question is which method do we use? The answer depends on whether one needs a quick or an accurate estimation. In what follows, the methods are ranked according to their accuracy or speed. The order of the methods based on their speed (computing time) are

1. Any graphical method.
2. Least Squares Method.
3. Maximum Likelihood Estimator.
4. Method of Moments.

The order of the methods based on their accuracy are

1. Applying the three analytical methods (MLE,MOM and LSM) and selecting the best one which gives the minimum mean squared error.
2. Method of Moments.
3. Maximum Likelihood Estimator.
4. Least Squares Method.
5. Any graphical method.

6. Conclusion

In this paper, we have presented both graphical and analytical methods for estimating the Weibull distribution parameters. It has been shown from the computational results that the method which gives the best estimates is the method of moments.

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