Multivariate Distribution Obtained by Hierarchical Mixing

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ABSTRACT

The object of this research is to investigate what are the probability distributions obtained by continuous averaging of parameters. Suppose we have the following univariate compoundings:

1. $x|\lambda$ has a Poisson distribution with parameter λ.
2. $\lambda|y$ has a gamma distribution with parameters h and y.
3. $y^{-1}|\mu$ has a gamma distribution with parameter μ^{-1} and k.

In this paper, I find the probability density functions of $\lambda|\mu$, $x|y$ and the probability mass function of $x|\mu$

1. Introduction

Suppose we have the following univariate compoundings:

(1) $x|\lambda$ has the probability mass function

$$P(x|\lambda) = \frac{e^{-\lambda}\lambda^{x}}{x!} \qquad x = 0,1,2,\ldots$$

(2) $\lambda|y$ has the probability density function

$$g_1(\lambda|y,h)) = \frac{(h/y)^h}{\Gamma(h)}\lambda^{h-1}e^{-(h/y)\lambda} \quad , \lambda>0$$

(3) $y^{-1}|\mu$ has the probability density function

$$g_2(y^{-1}|\mu^{-1},k) = \frac{(k/\mu^{-1})^k}{\Gamma(k)}(y^{-1})^{k-1}e^{-(k/\mu^{-1})y^{-1}} \quad , \; y^{-1}>0$$

There are three problems to be solved:

(1) What is the probability density function of $\lambda|\mu$?

(2) What is the probability mass function of $x|y$?

(3) What is the probability mass function of $x|\mu$?

In order to find more general results, I consider the following cases with parameters λ , μ , x , y:

	x	λ	y	μ	
Case 1	U	U	U	U	
Case 2	M	U	U	U	U: Univariate
Case 3	M	M	U	U	M: Multivariate
Case 4	M	M	M	M	

When referring to the joint distribution of n random variables, I use the term "joint generalized probability density function"(abbreviated by joint g.p.d.f.) to describe a function *f* that may be their joint probability mass function(p.m.f.), their joint probability density function(p.d.f.) or their joint mixed functions(p.m.f-p.d.f.).

The results for case 4 are trivial from Case1 for independence over i=1,2,…,n. The proof for each theorem is presented in Appendix A. Some properties of Confluent Hypergeometric Function are given in Appendix B.

2. Theoretical Results

The results for Case 1 are given in Theorems 1,2 and 3.

Theorem 1: The probability density function of λ | μ is

$$f_1(\lambda|\mu,h,k)=\int_0^\infty g_1(\lambda|h,y)g_2(y^{-1}|\mu^{-1},k)dy^{-1}$$

$$=\frac{\Gamma(h+k)(2h)^h(2k)^k(\lambda/\mu)^{h-1}\mu^{-1}}{\Gamma(h)\Gamma(k)\left[(2\lambda h/\mu)+2k\right]^{h+k}}\ ,\quad \lambda>0$$

Note: If we let λ'=(λ/μ), then λ' has an F distribution with 2h and 2k degrees of freedom. Therefore, λ has $\mu F_{2h,2k}$.

Theorem 2: The probability mass function of x | y is

$$f_2(x|y,h)=\int_0^\infty p(x|\lambda)g_1(\lambda|y,h)d\lambda$$

$$=\binom{x+h-1}{x}\left(\frac{y}{y+h}\right)^x\left(\frac{h}{y+h}\right)^h\ ,\quad x=0,1,2,\ldots$$

Note: This is a negative binomial distribution with parameters h and $\frac{y}{y+h}$.

Theorem 3: The probability mass function of $x|\mu$ is

$$f_3(x|\mu,h,k)=\int_0^\infty p(x|\lambda)f_1(\lambda|\mu,h,k)d\lambda$$

$$=\left(\frac{1}{\mu}\right)\binom{x+h-1}{x}\frac{\Gamma(h+k)}{\Gamma(k)}\left(\frac{k}{h}\right)^x U(x+h;x-k+1;\frac{k}{h})\ ,$$

x=0,1,2, ,x+h>0, $\frac{k}{h}>0$

Note: $U(x+h;x-k+1;\frac{k}{h})$ is a Confluent Hypergeometric Function (see Appendix B.3).

The results for Case 2 are given in Theorem 4 and 5.

Theorem 4: Let $\underline{x} = (x_1, x_2, ..., x_n)$, then the joint g.p.d.f. of $\underline{x}|y$ is

$$f_4(\underline{x}|y,h) = \int_0^\infty \left[\prod_{i=1}^{n} p(x_i|\lambda)\right] g_1(\lambda|h,y)d\lambda$$

$$= \binom{\sum_{i=1}^{n} x_i + h - 1}{\underline{x}, h-1} \left(\frac{y}{h+ny}\right)^{\sum_{i=1}^{n} x_i} \left(\frac{h}{h+ny}\right)^h,$$

$x_i = 0,1,2,\ldots$ for $i = 1,2,...,n$

Note: This is a multivariate negative binomial distribution with parameters $(h, \frac{y}{h+ny})$.

Theorem 5: Let $\underline{x} = (x_1, x_2, ..., x_n)$, then the joint g.p.d.f. of $\underline{x}|\mu$ is

$$f_5(\underline{x}|\mu,h,k) = \int_0^\infty \left[\prod_{i=1}^{n} p(x_i|\lambda)\right] f_1(\lambda|\mu,h,k)d\lambda$$

$$= (\frac{1}{\mu}) \binom{\sum_{i=1}^{n} x_i + h - 1}{\underline{x}, h-1} \frac{\Gamma(h+k)}{\Gamma(k)} (\frac{k}{h})^{\sum_{i=1}^{n} x_i} U(\sum_{i=1}^{n} x_i + h; \sum_{i=1}^{n} x_i - k + 1; \frac{nk}{h}),$$

$x_i = 0,1,2,\ldots$ for $i = 1,2,...,n$

The results for Case 3 are given in Theorems 6,7 and 8.

Theorem 6: Let $\underline{\lambda} = (\lambda_1, \lambda_2, ..., \lambda_n)$, $\underline{h} = (h_1, h_2, ..., h_n)$ then the joint g.p.d.f. of $\underline{\lambda}|\mu$ is

$$f_6(\underline{\lambda}|\underline{h},\mu,k) = \int_0^\infty \left[\prod_{i=1}^{n} g_1(\lambda_i|y,h_i)\right] g_2(y^{-1}|\mu^{-1},k)dy^{-1}$$

$$= \left[\prod_{i=1}^{n} \frac{(\lambda_i h_i)^{h_i}}{\lambda_i \Gamma(h_i)}\right] \left[\frac{(\mu k)^k}{\Gamma(k)}\right] \left\{ \frac{\Gamma(\sum_{i=1}^{n} h_i + k)}{\left[\sum_{i=1}^{n} h_i \lambda_i + \mu k\right]^{(\sum_{i=1}^{n} h_i + k)}} \right\}, \quad \lambda_i > 0 \text{ for } i = 1,2,...,n.$$

Corollary 6.1: If $h_i = h$ for $i = 1,2,...,n$, then

$$f_6^{'}(\underline{\lambda}\big|h,\mu,k) = \left(\prod_{i=1}^{n}\lambda_i\right)^{h-1}\left[\frac{h^h}{\Gamma(h)}\right]^n\left[\frac{(\mu k)^k}{\Gamma(k)}\right]\left\{\frac{\Gamma(nh+k)}{\left[h(\sum_{i=1}^{n}\lambda_i)+\mu k\right]^{nh+k}}\right\},$$

$\lambda_i > 0$ for $i = 1,2,...,n.$

Theorem 7: Suppose $\underline{\lambda}$ has the joint g.p.d.f. as given in Theorem 6. Define $\underline{\Phi} = \frac{\underline{\omega}}{(\nu/2k)}$

Where $\underline{\varpi} = (\frac{u_1}{2h_1},\frac{u_2}{2h_2},...,\frac{u_n}{2h_n})$. We obtain $\underline{\Phi}$ has $F_{2\underline{h},2k}$ distribution and the joint g.p.d.f. is

$$\left[\prod_{i=1}^{n}(\frac{h_i}{k})^{h_i}\right]B(k,\underline{h})^{-1}(\prod_{i=1}^{n}\phi_i^{h_i-1})(1+\sum_{i=1}^{n}\phi_i)^{-(\sum_{i=1}^{n}h_i+k)}$$

where $\phi_i = \frac{\lambda_i h_i}{k} > 0$ for $i = 1,2,...,n$ and $B(k,\underline{h})^{-1} = \frac{\Gamma(\sum_{i=1}^{n}h_i+k)}{\Gamma(k)\prod_{i=1}^{n}\Gamma(h_i)}$.

Furthermore, we have $\underline{\lambda} = \mu\underline{\Phi}$.

Corollary 7.1: If $h_i = h$ for $i = 1,2,...,n$ then $\underline{\lambda} = \mu F_{2h\underline{1},2k}$ where $\underline{1} = (1,1,...,1)$

Theorem 8: Let $\underline{x} = (x_1,x_2,...,x_n)$, $\underline{\lambda} = (\lambda_1,\lambda_2,...,\lambda_n)$, $\underline{h} = (h_1,h_2,...,h_n)$, then the joint g.p.d.f. of $\underline{x}\big|\mu$ is

$$f_7(\underline{x}\big|\underline{h},\mu,k) = \int_0^{\infty}....\int_0^{\infty}\left[\prod_{i=1}^{n}p(x_i\big|\lambda_i)\right]f_6(\underline{\lambda}\big|\underline{h},\mu,k)d\lambda_1...d\lambda_n$$

$$=\left[\prod_{i=1}^{n}\binom{h_i+x_i-1}{x_i}\right]\left[\frac{(\mu k)^k}{\Gamma(k)}\right]\frac{\Gamma(\sum_{i=1}^{n}h_i+k)}{\prod_{i=1}^{n}(h_i^{x_i})}$$

$$\cdot\left\{\prod_{i=1}^{n}U(x_i+h_i;x^{(i)}-h^{(i)}-(k-1);\frac{\left(\sum_{j=i+1}^{n}h_j\lambda_j\right)+\mu k}{h_i}\right\}$$

where $x_i = 0,1,2,\ldots$ for $i = 1,2,\ldots,n$

$$x^{(i)} = \sum_{j=1}^{i} x_j\ , \quad h^{(i)} = \sum_{j=i+1}^{n} h_j\ , \quad h^{(n)} = 0$$

Note: Because of $x^{(i)}$ and $h^{(i)}$ we have dependence.

Appendix A

(A.1) Proof of Theorem 1:

$$\begin{aligned}
f_1(\lambda \mid \mu,h,k) &= \int_0^\infty g_1(\lambda \mid h,y)\, g_2(y^{-1} \mid \mu^{-1},k)\, dy^{-1} \\
&= \int_0^\infty \frac{(h/y)^h}{\Gamma(h)} \lambda^{h-1} e^{-(\frac{h}{y})\lambda}\, \frac{(k/\mu^{-1})^k}{\Gamma(k)} (y^{-1})^{k-1} e^{-(\mu k)/y}\, dy^{-1} \\
&= \frac{(\mu k)^k h^h \lambda^{h-1}}{\Gamma(h)\Gamma(k)} \int_0^\infty (y^{-1})^{h+k-1} e^{-(\lambda h+\mu k)/y}\, dy^{-1} \\
&= \frac{(\mu k)^k h^h \lambda^{h-1}}{\Gamma(h)\Gamma(k)} \frac{\Gamma(h+k)}{(\lambda h+\mu k)^{h+k}} \\
&= \frac{\Gamma(h+k)(2h)^h (2k)^k}{\Gamma(h)\Gamma(k)} \mu^k \frac{\lambda^{h-1}}{(2\lambda h+2\mu k)^{h+k}} \\
&= (\frac{1}{\mu}) \frac{\Gamma(h+k)(2h)^h (2k)^k}{\Gamma(h)\Gamma(k)} \frac{(\lambda/\mu)^{h-1}}{(2h\frac{\lambda}{\mu}+2k)^{h+k}} \quad , \ \lambda > 0\ .
\end{aligned}$$

Let $\lambda' = \dfrac{\lambda}{\mu}$, we know

$$\frac{\Gamma(h+k)(2h)^h (2k)^k}{\Gamma(h)\Gamma(k)} \frac{(\lambda')^{h-1}}{(2h\lambda'+2k)^{h+k}}$$

is an F distribution with 2h and 2k degrees of freedom.

(A.2) Proof of Theorem 2:

$$\begin{aligned}
f_2(x \mid y,h) &= \int_0^\infty p(x \mid \lambda)\, g_1(\lambda \mid y,h)\, d\lambda = \int_0^\infty \frac{e^{-\lambda}\lambda^x}{x!} \frac{(h/y)^h}{\Gamma(h)} \lambda^{h-1} e^{-(h/y)\lambda}\, d\lambda \\
&= \frac{(h)^h}{y^h} \frac{1}{\Gamma(h)} \frac{1}{x!} \int_0^\infty \lambda^{x+h-1} e^{-(1+\frac{h}{y})\lambda}\, d\lambda = \frac{h^h}{y^h \Gamma(h) x!} \frac{\Gamma(x+h)}{(1+\frac{h}{y})^{x+h}}
\end{aligned}$$

$$=\frac{\Gamma(x+h)}{\Gamma(h)x!}\frac{h^h}{y^h}\frac{y^{x+h}}{(y+h)^{x+h}}=\binom{x+h-1}{x}(\frac{y}{y+h})^x(\frac{h}{y+h})^h\ , \qquad x=0,1,2,\ldots$$

(A.3) Proof of Theorem 3:

$$f_3(x|\mu,h,k)=\int_0^\infty p(x|\lambda)f_1(\lambda|\mu,h,k)d\lambda=\int_0^\infty\frac{e^{-\lambda}\lambda^x}{x!}\frac{1}{\mu}\frac{\Gamma(h+k)(2h)^h(2k)^k\lambda^{h-1}}{\Gamma(h)\Gamma(k)(2h\lambda+2k)^{h+k}}d\lambda$$

$$=\frac{\Gamma(h+k)(h)^h(k)^k}{\mu\cdot\Gamma(h)\Gamma(k)x!}\int_o^\infty\frac{e^{-\lambda}\lambda^{x+h-1}}{(h\lambda+k)^{h+k}}d\lambda$$

$$=\frac{\Gamma(h+k)(h)^h(k)^k}{\mu\Gamma(h)\Gamma(k)x!(h)^{h+k}}\int_0^\infty e^{-\lambda}\lambda^{x+h-1}(\lambda+\frac{k}{h})^{-(h+k)}d\lambda \qquad \text{(A.3.1)}$$

Since $U(a;b;x)=\frac{x^{1-b}}{\Gamma(a)}\int_0^\infty e^{-v}v^{a-1}(x+v)^{b-a-1}dv$, $a>0$ and x>0 (See Appendix B.3), comparing it with the integration part in (A.3.1), we know that $a=x+h$, $b=x-k+1$, $x=\frac{k}{h}$. Therefore,

$$\int_0^\infty e^{-\lambda}\lambda^{x+h-1}(\lambda+\frac{k}{h})^{-(h+k)}d\lambda=\frac{\Gamma(x+h)}{(\frac{k}{h})^{k-x}}\cdot U(x+h;x-k+1;\frac{k}{h}) \qquad \text{(A.3.2)}$$

Substituting (A.3.2) into (A.3.1), we obtain

$$f_3(x|\mu,h,k)=\frac{\Gamma(h+k)\Gamma(x+h)}{\mu\Gamma(h)\Gamma(k)x!}(\frac{k}{h})^x U(x+h;x-k+1;\frac{k}{h})$$

$$=\frac{1}{\mu}\binom{x+h-1}{x}\frac{\Gamma(h+k)}{\Gamma(k)}(\frac{k}{h})^x U(x+h;x-k+1;\frac{k}{h})$$

Where $x=0,1,2,\ldots,x+h>0$ and $\frac{k}{h}>0$.

(A.4) Proof of Theorem 4:

$$f_4(\underline{x}|y,h)=\int_0^\infty\left[\prod_{i=1}^n p(x_i|\lambda)\right]g_1(\lambda|y,h)d\lambda$$

$$=\int_0^\infty\frac{e^{-n\lambda}\lambda^{(\sum_{i=1}^n x_i)}}{\prod_{i=1}^n(x_i!)}\frac{(h/y)^h}{\Gamma(h)}\lambda^{h-1}e^{-(h/y)\lambda}d\lambda$$

$$=\frac{(h/y)^h}{\Gamma(h)\prod_{i=1}^{n}(x_i!)}\int_0^\infty \lambda^{(\sum_{i=1}^{n}x_i+h-1)} e^{-(n+\frac{h}{y})\lambda} d\lambda = \frac{(h/y)^h\,\Gamma(h+\sum_{i=1}^{n}x_i)}{\Gamma(h)\prod_{i=1}^{n}(x_i!)(n+\frac{h}{y})^{h+\sum_{i=1}^{n}x_i}}$$

$$=\binom{\sum_{i=1}^{n}x_i+h-1}{\underline{x},h-1}\left(\frac{y}{h+ny}\right)^{\sum_{i=1}^{n}x_i}\left(\frac{h}{h+ny}\right)^h$$

Where $x_i=0,1,2,\ldots$ for $i=1,2,\ldots,n$ and $\binom{\sum_{i=1}^{n}x_i+h-1}{\underline{x},h-1}=\frac{\Gamma(\sum_{i=1}^{n}x_i+h)}{\Gamma(h)\prod_{i=1}^{n}(x_i!)}$

(A.5) Proof of Theorem 5:

$$f_5(\underline{x}|\mu,h,k)=\int_0^\infty\left[\prod_{i=1}^{n}p(x_i|\lambda)\right]f_1(\lambda|\mu,h,k)d\lambda$$

$$=\int_0^\infty \frac{e^{-n\lambda}\lambda^{(\sum_{i=1}^{n}x_i)}}{\prod_{i=1}^{n}(x_i!)}\frac{1}{\mu}\frac{\Gamma(h+k)(2h)^h(2k)^k}{\Gamma(h)\Gamma(k)}\frac{\lambda^{h-1}}{(2h\lambda+2k)^{h+k}}d\lambda$$

$$=\frac{\Gamma(h+k)(2h)^h(2k)^k}{\mu\Gamma(h)\Gamma(k)\prod_{i=1}^{n}(x_i!)}\int_0^\infty\frac{\lambda^{(\sum_{i=1}^{n}x_i+h-1)}e^{-n\lambda}}{(2h\lambda+2k)^{h+k}}d\lambda \qquad \text{(A.5.1)}$$

In the integration part, let $n\lambda=t$, we obtain

$$\int_0^\infty\frac{\lambda^{(\sum_{i=1}^{n}x_i+h-1)}e^{-n\lambda}}{(2h\lambda+2k)^{h+k}}d\lambda=\int_0^\infty\frac{(\frac{t}{n})^{(\sum_{i=1}^{n}x_i+h-1)}e^{-t}}{(2h\frac{t}{n}+2k)^{h+k}}\frac{1}{n}dt$$

$$=\frac{(\frac{1}{n})^{(\sum_{i=1}^{n}x_i+h-1)}(\frac{1}{n})}{2^{h+k}(\frac{1}{n})^{h+k}}\int_0^\infty t^{(\sum_{i=1}^{n}x_i+h-1)}e^{-t}(ht+nk)^{-(h+k)}dt$$

$$=\frac{(\frac{1}{n})^{(\sum_{i=1}^{n}x_i-k)}}{(2h)^{h+k}}\int_0^\infty t^{(\sum_{i=1}^{n}x_i+h-1)}e^{-t}(t+\frac{nk}{h})^{-(h+k)}dt \qquad \text{(A.5.2)}$$

Comparing the integration part in (A.5.2) with $U(a;b;x)$ given in (B.3.2), we know $a=\sum_{i=1}^{n}x_i+h,\ \ b=\sum_{i=1}^{n}x_i-k+1,\ \ x=\frac{nk}{h}$. Therefore, (A.5.2) is equal to

$$\frac{(\frac{1}{n})^{(\sum_{i=1}^{n}x_i-k)}}{(2h)^{h+k}}\frac{\Gamma(\sum_{i=1}^{n}x_i+h)}{(\frac{nk}{h})^{k-\sum_{i=1}^{n}x_i}}\cdot U(\sum_{i=1}^{n}x_i+h;\sum_{i=1}^{n}x_i-k+1;\frac{nk}{h}) \qquad \text{(A.5.3)}$$

Substituting (A.5.3) into (A.5.1), we obtain

$$f_5(\underline{x},\Big|\mu,h,k)=(\frac{1}{\mu})\frac{\Gamma(h+k)}{\Gamma(h)\Gamma(k)\prod_{i=1}^{n}(x_i!)}(\frac{k}{h})^{\sum_{i=1}^{n}x_i}\Gamma(\sum_{i=1}^{n}x_i+h)$$

$$\cdot U(\sum_{i=1}^{n}x_i+h;\sum_{i=1}^{n}x_i-k+1;\frac{nk}{h})$$

$$=(\frac{1}{\mu})\left(\begin{matrix}\sum_{i=1}^{n}x_i+h-1\\ \underline{x},h-1\end{matrix}\right)\frac{\Gamma(h+k)}{\Gamma(k)}(\frac{k}{h})^{\sum_{i=1}^{n}x_i}\cdot U(\sum_{i=1}^{n}x_i+h;\sum_{i=1}^{n}x_i-k+1;\frac{nk}{h})$$

where $x_i=0,1,2,...,$ for $i=1,2,...,n$.

(A.6) Proof of Theorem 6:

$$f_6(\underline{\lambda}\big|\underline{h},\mu,k)=\int_0^\infty\left[\prod_{i=1}^{n}g_1(\lambda_i\big|y,h_i)\right]g_2(y^{-1}\big|\mu^{-1},k)dy^{-1}$$

$$=\int_0^\infty\left\{\prod_{i=1}^{n}\left[\frac{(h_i/y)^{h_i}}{\Gamma(h_i)}\lambda_i^{h_i-1}e^{-\lambda_i(h_i/y)}\right]\right\}\frac{(k/\mu^{-1})^k}{\Gamma(k)}(y^{-1})^{k-1}e^{-(k\mu/y)}dy^{-1}$$

$$=\int_0^\infty\frac{\prod_{i=1}^{n}(h_i)^{h_i}}{\prod_{i=1}^{n}\Gamma(h_i)}(\prod_{i=1}^{n}\lambda_i^{h_i-1})(y^{-1})^{\sum_{i=1}^{n}h_i}e^{(\sum_{i=1}^{n}h_i\lambda_i)y^{-1}}\cdot\frac{(\mu k)^k}{\Gamma(k)}(y^{-1})^{k-1}e^{-(\mu k/y)}dy^{-1}$$

$$= \frac{\prod_{i=1}^{n} (h_i)^{h_i}}{\prod_{i=1}^{n} \Gamma(h_i)} (\prod_{i=1}^{n} \lambda_i^{h_i - 1}) \frac{(\mu k)^k}{\Gamma(k)} \int_0^{\infty} (y^{-1})^{\sum_{i=1}^{n} h_i + k - 1} e^{-(\sum_{i=1}^{n} h_i \lambda_i + \mu k) y^{-1}} dy^{-1}$$

$$= \left[\prod_{i=1}^{n} \frac{(\lambda_i h_i)^{h_i}}{\lambda_i \Gamma(h_i)} \right] \left[\frac{\Gamma(\sum_{i=1}^{n} h_i + k)}{(\sum_{i=1}^{n} \lambda_i h_i + \mu k)^{\sum_{i=1}^{n} h_i + k}} \right] \frac{(\mu k)^k}{\Gamma(k)}, \text{ where } \lambda_i > 0 \text{ for } i = 1,2,...,n.$$

(A.7) Proof of Theorem 7:

Since $\lambda_i | y$ has the p.d.f. $g(\lambda_i | y, h_i) = \frac{(h_i / y)^{h_i}}{\Gamma(h_i)} \lambda_i^{h_i - 1} e^{-(h_i / y) \lambda_i}, \lambda_i > 0$.

We know that $E(\underline{\lambda} | y) = y \cdot \underline{1}$ where $\underline{1} = (1,1,...,1)$. Let χ_n^2 denote the Chi-square Distribution with n degrees of freedom, we know that $E(\chi_n^2) = n = 2h_i$.

Therefore, $\frac{\underline{\lambda}}{y} | y$ has the following coordinates

$$\frac{\lambda_i}{y} \bigg| y \sim \frac{\chi_{2h_i}^2}{2h_i} (= \frac{u_i}{2h_i}) \text{ for } i = 1,2,...,n.$$

Since $y^{-1} | \mu$ has the p.d.f. $g_2(y^{-1} | \mu^{-1}, k)$, we know that $E(y^{-1}) = \mu^{-1}$. Therefore, $\frac{\mu}{y} \sim \frac{\chi_{2k}^2}{2k} (= \frac{v}{2k})$. Hence, $\frac{\underline{\lambda} / y}{\mu / y}$ has the following coordinates

$$\frac{\lambda_i / y}{\mu / y} = \frac{\lambda_i}{\mu} = \frac{u_i / 2h_i}{v / 2k} \text{ for } i = 1,2,...,n.$$

Define $\underline{\Phi} = \frac{\underline{\omega}}{(v/2k)}$ where $\underline{\omega} = (\frac{u_1}{2h_1}, \frac{u_2}{2h_2}, ..., \frac{u_n}{2h_n})$, then from the g.p.d.f. of $\underline{\lambda}$ given in Theorem 6, we know $\underline{\Phi} \sim F_{2\underline{h}, 2k}$ and $\underline{\Phi}$ has the g.p.d.f.

$\left[\prod_{i=1}^{n} (\frac{h_i}{k})^{h_i} \right] B(k, \underline{h})^{-1} (\prod_{i=1}^{n} \phi_i^{h_i - 1})(1 + \sum_{i=1}^{n} \phi_i)^{-(\sum_{i=1}^{n} h_i + k)}$, where $\phi_i = \frac{\lambda_i h_i}{k} > 0$ for $i = 1,2,...,n$ and

$B(k, \underline{h})^{-1} = \frac{\Gamma(\sum_{i=1}^{n} h_i + k)}{\Gamma(k) \prod_{i=1}^{n} \Gamma(h_i)}$. Furthermore, we obtain $\underline{\lambda} = \mu \cdot \underline{\Phi}$.

(A.8) Proof of Theorem 8:

$$f_7(\underline{x}|\underline{h},\mu,k)=\int_0^\infty\cdots\int_0^\infty\left[\prod_{i=1}^n p(x_i|\lambda_i)\right]f_6(\underline{\lambda}|\underline{h},\mu,k)d\lambda_1\cdots d\lambda_n$$

$$=\frac{(\prod_{i=1}^n h_i^{h_i})(\mu k)^k\,\Gamma(\sum_{i=1}^n x_i+k)}{\left[\prod_{i=1}^n\Gamma(h_i)\right]\Gamma(k)\left[\prod_{i=1}^n(x_i!)\right]}\int_0^\infty\cdots\int_0^\infty\frac{\left[\prod_{i=1}^n(\lambda_i)^{x_i+h_i-1}\right]e^{-\sum_{i=1}^n\lambda_i}}{(\sum_{i=1}^n\lambda_i h_i+\mu k)^{\sum_{i=1}^n h_i+k}}d\lambda_1\cdots d\lambda_n \qquad \text{(A.8.1)}$$

In the multiple integration of (A.8.1), if we integrate with respect to λ_1 first, then we have

$$\int_0^\infty\cdots\left[\int_0^\infty\frac{\lambda_1^{x_1+h_1-1}\prod_{i=2}^n(\lambda_i^{x_i+h_i-1})e^{-\lambda_1}e^{-\sum_{i=2}^n\lambda_i}}{(h_1\lambda_1+\mu k+\sum_{i=2}^n h_i\lambda_i)^{(h_1+k+\sum_{i=2}^n h_i)}}d\lambda_1\right]d\lambda_2\cdots d\lambda_n$$

$$=\int_0^\infty\cdots\int_0^\infty\frac{\prod_{i=2}^n(\lambda_i^{x_i+h_i-1})e^{-\sum_{i=2}^n\lambda_i}}{(h_1)^{\sum_{i=1}^n h_i+k}}\left\{\int_0^\infty\frac{\lambda_1^{x_1+h_1-1}e^{-\lambda_1}}{[\lambda_1+(\frac{\mu k+\sum_{i=2}^n h_i\lambda_i}{h_1})]^{(\sum_{i=1}^n h_i+k)}}d\lambda_1\right\}d\lambda_2\cdots d\lambda_n \qquad \text{(A.8.2)}$$

Comparing the integration inside the braces of (A.8.2) with $U(a;b;x)$ given in (B.3.2), we know

$$a=x_1+h_1,\ b=x_1-(\sum_{i=2}^n h_i+k)+1,\ x=(\sum_{i=2}^n h_i\lambda_i+\mu k)/h_1\ .$$

Therefore, $$\int_0^\infty\frac{\lambda_1^{x_1+h_1-1}e^{-\lambda_1}}{[\lambda_1+(\frac{\mu k+\sum_{i=2}^n h_i\lambda_i}{h_1})]^{(\sum_{i=1}^n h_i+k)}}d\lambda_1$$

$$=\frac{\Gamma(x_1+h_1)}{\left(\frac{\sum_{i=2}^n h_i\lambda_i+\mu k}{h_1}\right)^{\sum_{i=2}^n h_i+k-x_1}}\cdot U(x_1+h_1;x_1-(\sum_{i=2}^n h_i+k)+1;\frac{\sum_{i=2}^n h_i\lambda_i+\mu k}{h_1}) \qquad \text{(A.8.3)}$$

Substituting (A.8.3) into (A.8.2), we obtain

$$\frac{\Gamma(x_1+h_1)}{(h_1)^{x_1+h_1}}\cdot U(x_1+h_1;x_1-\sum_{i=2}^{n}h_i-(k-1);\frac{\sum_{i=2}^{n}h_i\lambda_i+\mu k}{h_1})$$

$$\cdot\int_0^\infty\cdots\int_0^\infty\prod_{i=2}^{n}(\lambda_i)^{x_i+h_i-1}e^{-\sum_{i=2}^{n}\lambda_i}(\sum_{i=2}^{n}h_i\lambda_i+\mu k)^{x_1-(\sum_{i=2}^{n}h_i+k)}d\lambda_2\cdots d\lambda_n \qquad \text{(A.8.4)}$$

In the multiple integration of (A.8.4), we now integrate with respect to λ_2,

$$\int_0^\infty\cdots\int_0^\infty\prod_{i=3}^{n}(\lambda_i)^{x_i+h_i-1}e^{-\sum_{i=3}^{n}\lambda_i}\left[\int_0^\infty e^{-\lambda_2}\lambda_2^{x_2+h_2-1}(\sum_{i=2}^{n}h_i\lambda_i+\mu k)^{x_1-(\sum_{i=2}^{n}h_i+k)}d\lambda_2\right]d\lambda_3\cdots d\lambda_n \quad \text{(A.8.5)}$$

Comparing the integration inside the bracket of (A.8.5) with $U(a;b;x)$, we know

$a=x_2+h_2$, $b=(x_1+x_2)-\sum_{i=3}^{n}h_i-(k-1)$, $x=\dfrac{\sum_{i=3}^{n}h_i\lambda_i+\mu k}{h_2}$. Hence, the integration with respective to λ_2 is equal to

$$\frac{\Gamma(x_2+h_2)(h_2)^{x_1-(\sum_{i=2}^{n}h_i+k)}}{\left(\dfrac{\sum_{i=2}^{n}h_i\lambda_i+\mu k}{h_2}\right)^{\sum_{i=3}^{n}h_i+k-(x_1+x_2)}}\cdot U(x_2+h_2;(x_1+x_2)-\sum_{i=3}^{n}h_i-(k-1);\frac{\sum_{i=3}^{n}h_i\lambda_i+\mu k}{h_2}) \quad \text{(A.8.6)}$$

Substituting (A.8.6) into (A.8.5), we obtain

$$\frac{\Gamma(x_2+h_2)}{h_2^{x_2+h_2}}U(x_2+h_2;(x_1+x_2)-\sum_{i=3}^{n}h_i-(k-1);\frac{\sum_{i=3}^{n}h_i\lambda_i+\mu k}{h_2})$$

$$\cdot\int_0^\infty\cdots\int_0^\infty\prod_{i=3}^{n}(\lambda_i)^{x_i+h_i-1}e^{-\sum_{i=3}^{n}\lambda_i}(\sum_{i=3}^{n}h_i\lambda_i+\mu k)^{(x_1+x_2)-(\sum_{i=3}^{n}h_i+k)}d\lambda_3\cdots d\lambda_n$$

By doing the integration step by step, we finally obtain $f_7(\underline{x}|\underline{h},\mu,k)$ given in Theorem 8.

Appendix B

(B.1) Generalized Hypergeometric Functions:

The generalized hypergeometric differential equation is

$$\left\{x\frac{d}{dx}(x\frac{d}{dx}+b_1-1)(x\frac{d}{dx}+b_2-1)\cdots(x\frac{d}{dx}+b_B-1)-x(x\frac{d}{dx}+a_1)(x\frac{d}{dx}+a_2)\cdots(x\frac{d}{dx}+a_A)\right\}y=0$$

(B.1.1)

This is a linear equation of order max(A,B+1), and its solution is in terms of the generalized hypergeometric function:

$${}_AF_B[a_1,a_2,...,a_A;b_1,b_2,...,b_B;x]={}_AF_B[(a);(b);x]=\sum_{n=0}^{\infty}\frac{(a_1)_n(a_2)_n\cdots(a_A)_n x^n}{(b_1)_n(b_2)_n\cdots(b_B)_n n!}=\sum_{n=0}^{\infty}\frac{((a)_n)x^n}{((b)_n)n!} \quad \text{(B.1.2)}$$

Where $(a)_n = a(a+1)(a+2)\cdots(a+n-1)$ and $(a)_0 = 1$, for all a_r, b_r and x can be real or complex number provided that no b is a negative integer (Bailey, 1935, Chapter 2). Here x is called the "variable' and a_r, b_r are called the "parameters" of the function.
In particular, if A=B=1, then the equation (B.1.1) reduces to

$$x\frac{d^2y}{dx^2}+(b-x)\frac{dy}{dx}-ay=0 \quad \text{(B.1.3)}$$

This equation is the Confluent Hypergeometric Equation known as Kummer's equation, and any solution of equation (B.1.3) is a Confluent Hypergeometric Function. The simplest solution is Kummer's function,

$${}_1F_1[a;b;x]=\sum_{n=0}^{\infty}\frac{(a)_n x^n}{(b)_n n!} \quad \text{(B.1.4)}$$

This is the standard hypergeometric notation introduced by Pochhammer in 1870 and modified by Barnes (1908, a,b).

(B.2) The Second Form of Solutions of Kummer's Equation:

Define an alternative form of solution of Kummer's equation as

$$U(a;b;x)=\frac{\Gamma(1-b)}{\Gamma(1+a-b)}{}_1F_1[a;b;x]+\frac{\Gamma(b-1)}{\Gamma(a)}x^{1-b}{}_1F_1[1+a-b;2-b;x] \quad \text{(B.2.1)}$$

The function $U(a;b;x)$ is a many-valued function of x. This function is analytic for all values of a,b and x real or complex number even when b is zero or a negative integer.

(B.3) Euler Type Integrals for $U(a;b;x)$:

The corresponding Euler integral for $U(a;b;x)$ is given by

$$U(a;b;x)=\frac{1}{\Gamma(a)}\int_0^{\infty}e^{-xt}t^{a-1}(1+t)^{b-a-1}dt \ \text{ for } a>0,\ x>0 \quad \text{(B.3.1)}$$

This integral can also be transformed in various ways, for example, if $tx = v$, then

$$U(a;b;x)=\frac{x^{1-b}}{\Gamma(a)}\int_0^{\infty}e^{-v}v^{a-1}(x+v)^{b-a-1}dv \quad \text{(B.3.2)}$$

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