CONFIDENCE BANDS FOR GENERAL LINEAR MODELS WITH RESTRICTED INDEPENDENT VARIABLES

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ABSTRACT

This article presents a method to find the critical points for constructing exact $(1-\alpha)100\%$ confidence bands for a General Linear Model over certain subsets of the predictor space. An application of this procedure in Response Surface Methodology is described, tables of critical values are also included.

1. INTRODUCTION

Scheffé (1959) developed the procedure for constructing a confidence band for regression functions with unrestricted independent variables. However, Scheffé's result is conservative in the sense that we are usually not interested in the entire predictor space, but in a subset of it. In many cases, it is possible to construct a narrower confidence band if the subset is of a particular form.

Consider the model

$$Y = X\beta + \epsilon \tag{1.1}$$

where Y is an $n \times 1$ vector of observations, X is the nxp full rank design matrix, β is a $p \times 1$ vector of unknown regression parameters and ϵ is the $n \times 1$ vector of random errors. The usual assumption made is that $\epsilon \sim N(0, \sigma^2 I)$. Under this assumption, $\hat{\beta}$ is distributed as $N(\beta, \sigma^2(X'X)^{-1})$, where $\hat{\beta}$ is the least square estimator of β. A restricted confidence band for (1.1) can be defined as a set of the following form:

$$\{\beta : (\mathrm{x}'\hat{\beta} - \mathrm{x}'\beta)^2 \leq c^2 p s^2 \mathrm{x}'(X'X)^{-1}\mathrm{x},\ \forall \mathrm{x} \in C_{\mathbf{x}}\},$$

where p is the number of model parameters, c^2 is a specified constant, s^2 is the error mean square with $\nu s^2/\sigma^2$ distributed as χ^2_ν independent of $\hat{\beta}$, and $C_{\mathbf{x}}$ is the subset of R^p. If $(X'X)^{-1} = PDP'$, where P is a matrix consisting of the orthonormal eigenvectors of $(X'X)^{-1}$ and D is the diagonal matrix consisting of its eigenvalues, let $W = XPD^{1/2}$, and $\beta_{\mathbf{w}} = D^{-1/2}P'\beta$. Then model (1.1) can be transformed to

$$Y = W\beta_{\mathbf{w}} + \epsilon. \tag{1.2}$$

Since $W'W = I$, the least square estimator $\hat{\beta}_{\mathbf{w}}$ is distributed as $N(\beta_{\mathbf{w}}, \sigma^2 I)$. Note that the $\hat{\beta}_{\mathbf{w}}$ are now independent normal random variables, with a common constant variance. This makes the problem more tractable. A restricted confidence band for (1.2) can be defined as a set of the following form:

$$\{\beta_{\mathbf{w}} : (\mathrm{w}'\hat{\beta}_{\mathbf{w}} - \mathrm{w}'\beta_{\mathbf{w}})^2 \leq c^2 p s^2 \mathrm{w}'\mathrm{w},\ \forall \mathrm{w} \in C^*_{\mathbf{w}}\}, \tag{1.3}$$

where $\mathrm{w} = D^{1/2}P'\mathrm{x}$, $\beta_{\mathbf{w}} = D^{-1/2}P'\beta$. Many researchers have derived confidence bands for different forms of restrictions. Bohrer (1967) gave sharp confidence bounds over the region

$C_{\mathbf{w}}^{+} = \{\mathbf{w} : w_i \geq 0,\ i = 1, ..., p\}$. Bohrer and Francis (1972a) developed one-sided bounds for the region $C_{\mathbf{x}}^{+} = \{\mathbf{x} : x_i \geq 0,\ i = 1, ..., p\}$. Casella and Strawderman (1980) focused on $C_{\mathbf{w}}^{*} = \{\mathbf{w} : \sum_{i=1}^{r} \mathbf{w}_i^2 \geq q_i^2 \sum_{i=r+1}^{p} \mathbf{w}_i^2\}$. Inference on linear regression are discussed by Wynn and Bloomfield (1971), and Bohrer and Francis (1972b). Determine lower bounds for coverage probability of a confidence band are considered by Halperin, Rostogi, Ho and Yang (1967) and Halperin and Gurian (1968).

We focus here on sets $C_{\mathbf{w}}$ that have the form

$$C_{\mathbf{w}} = \{\mathbf{w} : (\mathbf{w}_1'\mathbf{w}_1)^2 = q^2(\mathbf{w}_2'\mathbf{w}_2) = r^4\}, \tag{1.4}$$

where r, q are positive constants, $\mathbf{w}' = (\mathbf{w}_1', \mathbf{w}_2')$ with $\mathbf{w}_1$ k-variate and $\mathbf{w}_2$ s-variate (with $k + s = p$).

Although this set may seem restrictive, an example of an application is provided in section 4. In section 2 of this article, we derive an exact confidence band for (1.2) restricted to $C_{\mathbf{w}}$ of (1.4). In section 3 a computational formula is derived in order to numerically evaluate the coverage probabilities.

2. THE THEORETICAL SOLUTION

Theorem 1:

Let $S(C_{\mathbf{w}}) = \{\beta_{\mathbf{w}} : (\mathbf{w}'\hat{\beta}_{\mathbf{w}} - \mathbf{w}'\beta_{\mathbf{w}})^2 \leq c^2 p s^2 \mathbf{w}'\mathbf{w},\ \forall \mathbf{w} \in C_{\mathbf{w}}\}$, where s^2 is an estimator of σ^2 which satisfies $\nu s^2/\sigma^2 \sim \chi_\nu^2$ independent of $\hat{\beta}_{\mathbf{w}} \sim N_p(\beta_{\mathbf{w}}, \sigma^2 I)$. Then,

$$P(S(C_{\mathbf{w}})) = P\{(\chi_k + \tfrac{r}{q}\chi_s)^2 \leq \tfrac{c^2}{\nu} p\, \chi_\nu^2 (1 + (r^2/q^2)\}, \tag{2.1}$$

where r and q are the constants as defined in set $C_{\mathbf{w}}$, and χ_k and χ_s are the square roots of independent χ^2 random variables with degrees of freedom k and s, respectively, with $k + s = p$ and independent of χ_ν^2.

3. COMPUTATIONAL EXPRESSIONS

In this section, we provide a computational formula for (2.1) since this coverage probability is quite difficult to evaluate numerically. To derive the computational formula, two preliminary lemmas are required.

Lemma 1 :

$P((\chi_k + \frac{r}{q}\chi_s)^2 \leq \frac{c^2}{\nu} p\, \chi^2_\nu(1 + \frac{r^2}{q^2}),\, \chi^2_k + \chi^2_s < \frac{c^2}{\nu} p\, \chi^2_\nu) = P(F_{p,\nu} \leq c^2)$

where $F_{p,\nu}$ is an F random variable with p and ν degrees of freedom.

Lemma 2 :

$$P((\chi_k + \tfrac{r}{q}\chi_s)^2 \leq \tfrac{c^2}{\nu} p\, \chi^2_\nu(1 + \tfrac{r^2}{q^2}),\, \chi^2_k + \chi^2_s \geq \tfrac{c^2}{\nu} p\, \chi^2_\nu) \tag{3.1}$$

$$= P(0 \leq V \leq \lambda_2,\, c^2 \leq F_{p,\nu} \leq c^2(1 + (q^2/r^2)))$$

$$+P(V \geq \lambda_1,\, c^2 \leq F_{p,\nu} \leq c^2(1 + (r^2/q^2)))$$

where $V = \chi_k/\chi_s$, $\lambda_1 = \lambda_1(F_{p,\nu}) = (-\frac{r}{q}F_{p,\nu} - (1 + \frac{r^2}{q^2})\sqrt{F_{p,\,\nu}c^2 - c^4})/(F_{p,\nu} - c^2(1 + \frac{r^2}{q^2}))$ and $\lambda_2 = \lambda_2(F_{p,\nu}) = (-\frac{r}{q}F_{p,\nu} + (1 + \frac{r^2}{q^2})\sqrt{F_{p,\,\nu}c^2 - c^4})/(F_{p,\nu} - c^2(1 + \frac{r^2}{q^2}))$.

Theorem 2 :

$$P\{(\chi_k + \tfrac{r}{q}\chi_s)^2 \leq \tfrac{c^2}{\nu} p\, \chi^2_\nu(1 + (r^2/q^2)\}$$

$$= P(F_{p,\nu} < c^2)$$

$$+\int_{c^2}^{c^2(1+d^{-2})} P(F_{k,s} \leq \tfrac{s}{k}\lambda_2^2(t))dG(t)$$

$$+\int_{c^2}^{c^2(1+d^2)} P(F_{k,s} \geq \tfrac{s}{k}\lambda_1^2(t))dG(t)$$

where $d^2 = r^2/q^2$, $G(t)$ is the CDF of $F_{p,\nu}$ and $F_{k,s}$ is an F random variable with k and s degrees of freedom.

Table I and II give values of c^2 for which the coverage probability is 0.90 for selected values of d^2, k, p, and ν. They were obtained using a Fortran program which calls the Engineering and Scientific Subroutine Library (ESSL(1990)) procedure DGLNQ which performed Gaussian Quadrature using 64 points. We also make use of a subroutine written by R.J.

Craig (1984) which calculates cumulative F-distribution probabilities. (Our program is available upon request from the second author.)

4. AN APPLICATION

The procedure can be used in Response Surface Methodology. As an application, we consider a second-order model that is both rotatable and orthogonal. A model is said to be rotatable if the variance at a particular point is dependent only on its distance from the center, not on its orientation. A model is said to be orthogonal if the variance-covariance matrix is diagonal.

A rotatable design must have design points, $\mathbf{x}'_u = (x_{1u},\ x_{2u}, ...,\ x_{ku})$, for $u = 1,\ ...,\ n$, which satisfy the following requirements (Box and Draper, 1987; all summations are for $u = 1, ..., n$) :

1. $\sum\ x_{iu}^2 = n\lambda_3$, for each $i = 1, ..., k$,

2. $\sum\ x_{iu}^4 = 3\ \sum\ x_{iu}^2 x_{ju}^2 = 3n\lambda_4$, for each $1 \le i \ne j \le k$,

3. $\sum\ x_{hu}^s x_{iu}^r x_{ju}^m x_{tu}^f = 0$, for all other $s + r + m + f \le 4$,

for some constants λ_3, λ_4. A design that is both rotatable and orthogonal must satisfy one further requirement:

4. $\lambda_4/\lambda_3^2 = 1$.

Consider the following second order model:

$$Y = \beta_0 + \sum_{i=1}^k \beta_i x_i + \sum_{i=1}^k \beta_{ii} x_i^2 + \sqrt{2} \sum_{i<j} \beta_{ij} x_i x_j.$$

Then the form of the mean response $E(Y|\mathbf{u})$ as a function of $\mathbf{u} = (x_1, ..., x_k,\ x_1^2,\ ...,\ x_k^2,\ \sqrt{2}x_1x_2,\ ...,\ \sqrt{2}x_{k-1}x_k)'$ is $E(Y|\mathbf{u}) = \beta_0 + \delta(\mathbf{u}) + \epsilon$, where $\delta(\mathbf{u}) = \sum_{i=1}^k \beta_i x_i + \sum_{i=1}^k \beta_{ii} x_i^2 + \sqrt{2} \sum_{i<j} \beta_{ij} x_i x_j$. If $\mathbf{u} = 0$, which may represent the current or control conditions, we may

want to compare the mean response at a specified $\mathbf{u}$ to the mean response at $\mathbf{u} = 0$ for all $\mathbf{u}$ in a specified set $C_{\mathbf{u}}$. This is called Multiple Comparisons with a Control in Response Surface Methodology (Sa and Edwards, 1993).

Suppose we want to obtain simultaneous confidence intervals for $\delta(\mathbf{u}) = E(Y|\mathbf{u}) - E(Y|0) = \mathbf{u}'\beta_{\mathbf{u}}$, where $\beta_{\mathbf{u}} = (\beta_1, \ldots, \beta_k, \beta_{11}, \ldots, \beta_{kk}, \beta_{12}, \ldots\ , \beta_{k-1,k})'$ for all $\mathbf{u} \in C_{\mathbf{u}}$, $C_{\mathbf{u}} = \{\mathbf{u} : (\mathbf{u}_1'\mathbf{u}_1)^2 = (\mathbf{u}_2'\mathbf{u}_2) = r_0^2\}$, where r_0^2 is a prespecified constant, and we partition $\mathbf{u}' = (\mathbf{u}_1', \mathbf{u}_2')$, so that $\mathbf{u}_1' = (x_1, \ldots, x_k)$ and $\mathbf{u}_2' = (x_1^2, \ldots, x_k^2, \sqrt{2}x_1x_2, \ldots, \sqrt{2}x_{k-1}x_k)$. The critical points ($c^2$) calculated in section 3 may be used here to derive a solution.

Let $\hat{\beta}_{\mathbf{u}}$ be an estimator with multivariate normal distribution with mean $\beta_{\mathbf{u}}$ and covariance matrix $\sigma^2 V$, V known, independent of an error mean square s^2 satisfying $\nu s^2/\sigma^2 \sim \chi^2_\nu$. The estimator of $\delta(\mathbf{u})$ is $\hat{\delta}(\mathbf{u}) = \mathbf{u}'\hat{\beta}_{\mathbf{u}}$ and its estimated standard error is $S(\mathbf{u}) = s(\mathbf{u}'V\mathbf{u})^{1/2}$. If the design is rotatable, Sa and Edwards (1993) show that V is a block-diagonal matrix of the following form:

$$V = \begin{pmatrix} aI_k & 0 & 0 \\ 0 & bI_k + cJ_k & 0 \\ 0 & 0 & bI_{k(k-1)/2} \end{pmatrix}$$

with $a = 1/n\lambda_3$, $b = 1/(2n\lambda_4)$, and $c = (\lambda_3^2 - \lambda_4)/(2n\lambda_4((k+2)\lambda_4 - \lambda_3^2))$, I_k and J_k being the kxk identity matrix and the kxk matrix of 1's respectively. If the design is both rotatable and orthogonal, $\lambda_4/\lambda_3^2 = 1$, then $c = 0$. Thus,

$$V = \begin{pmatrix} aI_k & 0 \\ 0 & bI_{k(k-1)/2} \end{pmatrix}.$$

This design provides $S(\mathbf{u}) = s(a\mathbf{u}_1'\mathbf{u}_1 + b\mathbf{u}_2'\mathbf{u}_2)^{1/2}$.

Theorem 3:

If the design is rotatable and orthogonal, $\hat{\delta}(\mathbf{u}) \pm c_\alpha s(\mathbf{u})$ are simultaneous $(1-\alpha)100\%$ confidence interval for $\delta(\mathbf{u})$ for $\mathbf{u} \in C_{\mathbf{u}}$, where c_α is the critical point in this article under a given α with ν degrees of freedom, $q^2 = a^2/b$ and $r^4 = a^2 r_0^2$.

5. CONCLUSION

Although the set of interest may seem restrictive, an example of an application was provided in section 4. Moreover, this set may be used as a frame for a more complicated set of interest. A substantial amount of work remains to be done. Ongoing work is to seek an exact solution for less restrictive sets of interest. The authors are pursuing these and related research avenues and invite participation and comment.

TableI. $c^2, \ k = 2, \ p = 5, \ \alpha = 0.1$

	$d^2 = r^2/q^2$			
ν	0.1	0.5	1.0	10.0
1	35.491	47.805	51.313	45.600
2	5.960	7.821	8.355	7.511
3	3.476	4.490	4.782	4.330
4	2.689	3.437	3.653	3.324
5	2.315	2.938	3.117	2.846
6	2.098	2.649	2.808	2.569
7	1.958	2.461	2.607	2.390
8	1.860	2.330	2.466	2.265
9	1.788	2.233	2.362	2.172
10	1.733	2.159	2.282	2.101
12	1.653	2.053	2.168	2.000
14	1.599	1.980	2.090	1.930
16	1.560	1.927	2.033	1.880
18	1.530	1.887	1.990	1.842
20	1.507	1.856	1.957	1.812
30	1.439	1.766	1.859	1.725
60	1.375	1.680	1.767	1.643

TableII. c^2, $k=3$, $p=9$, $\alpha=0.1$

		$d^2 = r^2/q^2$		
ν	0.1	0.5	1.0	10.0
1	34.940	49.723	54.722	52.072
2	5.617	7.840	8.593	8.204
3	3.191	4.398	4.807	4.600
4	2.426	3.313	3.614	3.464
5	2.063	2.798	3.048	2.925
6	1.853	2.501	2.720	2.613
7	1.717	2.307	2.507	2.410
8	1.622	2.172	2.358	2.268
9	1.552	2.072	2.248	2.163
10	1.498	1.994	2.163	2.082
12	1.420	1.884	2.041	1.966
14	1.368	1.808	1.957	1.887
16	1.329	1.753	1.896	1.829
18	1.300	1.711	1.850	1.785
20	1.278	1.679	1.814	1.751
30	1.212	1.583	1.708	1.651
60	1.149	1.492	1.607	1.555

APPENDIX

Proof of Theorem 1:

Let $\theta = \frac{1}{\sigma}(\hat{\beta}_{\mathbf{w}} - \beta_{\mathbf{w}}) = (\theta_1', \theta_2')'$ be partitioned according to $C_{\mathbf{w}}$, then θ_1 is $k \times 1$ and θ_2 is $s \times 1$ with $s = p - k$ and $\theta_1'\theta_1 \sim \chi_k^2$, a χ^2 random variable with k degrees of freedom, and $\theta_2'\theta_2 \sim \chi_s^2$, independent of $\theta_1'\theta_1$, and $\theta'\theta \sim \chi_p^2$.

$$P(S(C_{\mathbf{w}})) = P(\max_{\mathbf{w}\in C_{\mathbf{w}}} \frac{(\mathbf{w}'\hat{\beta}_{\mathbf{w}} - \mathbf{w}'\beta_{\mathbf{w}})^2}{ps^2\mathbf{w}'\mathbf{w}} \leq c^2)$$

$$= P(\max_{\mathbf{w}\in C_{\mathbf{w}}} \frac{(\mathbf{w}'\theta)^2}{(p/\nu)\chi_\nu^2\mathbf{w}'\mathbf{w}} \leq c^2)$$

$$= P(\max_{\mathbf{w}\in C_{\mathbf{w}}} \frac{(\mathbf{w}'\theta)^2}{\mathbf{w}'\mathbf{w}} \leq \frac{c^2}{\nu}p\ \chi_\nu^2)$$

Since $(\mathbf{w}'\theta)^2 = (\mathbf{w}_1'\theta_1 + \mathbf{w}_2'\theta)^2 = (|\mathbf{w}_1||\theta_1| + |\mathbf{w}_2||\theta_2|)^2$ (note $|.|$ denotes the Euclidean norm) if and only if $\mathbf{w}_1 = c_1\theta_1$ and $\mathbf{w}_2 = c_2\theta_2 (c_1,\ c_2 > 0)$, by the Cauchy-Schwarz inequality. Then,

$$P(S(C_{\mathbf{w}})) = P(\max_{\mathbf{w}=(c_1\theta_1', c_2\theta_2')'\in C_{\mathbf{w}}} \frac{(|\mathbf{w}_1||\theta_1| + |\mathbf{w}_2||\theta_2|)^2}{\mathbf{w}'\mathbf{w}} \leq \frac{c^2}{\nu}p\ \chi_\nu^2)$$

$$= P(\max_{\mathbf{w}=(c_1\theta_1', c_2\theta_2')'\in C_{\mathbf{w}}} \frac{(|\mathbf{w}_1||\theta_1| + |\mathbf{w}_2||\theta_2|)^2}{\mathbf{w}'\mathbf{w}} \leq \frac{c^2}{\nu}p\ \chi_\nu^2,\ \theta \in C_{\mathbf{w}})$$

$$+ P(\max_{\mathbf{w}=(c_1\theta_1', c_2\theta_2')'\in C_{\mathbf{w}}} \frac{(|\mathbf{w}_1||\theta_1| + |\mathbf{w}_2||\theta_2|)^2}{\mathbf{w}'\mathbf{w}} \leq \frac{c^2}{\nu}p\ \chi_\nu^2,\ \theta \notin C_{\mathbf{w}})$$

If $\theta \in C_{\mathbf{w}}$, then $\max_{\mathbf{w}=(c_1\theta_1', c_2\theta_2')'\in C_{\mathbf{w}}} \frac{(\mathbf{w}'\theta)^2}{\mathbf{w}'\mathbf{w}} = \theta'\theta$ (ie. $\mathbf{w} = \theta$).

If $\theta \notin C_{\mathbf{w}}$, take $\mathbf{w}' = (\frac{r}{|\theta_1|}\theta_1',\ \frac{r^2}{q|\theta_2|}\theta_2')$ and since $\mathbf{w}'\mathbf{w} = (\mathbf{w}_1'\mathbf{w}_1 + \mathbf{w}_2'\mathbf{w}_2) = r^2 + r^4/q^2$, then $\max_{\mathbf{w}=(c_1\theta_1', c_2\theta_2')'\in C_{\mathbf{w}}} \frac{(\mathbf{w}'\theta)^2}{\mathbf{w}'\mathbf{w}} = (r|\theta_1| + \frac{r^2}{q}|\theta_2|)^2/(r^2 + \frac{r^4}{q^2})$.

Now, $P(S(C_{\mathbf{w}})) = P(\theta'\theta \leq \frac{c^2}{\nu}p\ \chi_\nu^2,\ \theta \in C_{\mathbf{w}}) + P((r|\theta_1| + \frac{r^2}{q}|\theta_2|)^2 \leq \frac{c^2}{\nu}p\ \chi_\nu^2(r^2 + \frac{r^4}{q^2}),\ \theta \notin C_{\mathbf{w}})$. Since $\theta \sim N_p(0,\ \mathbf{I})$, it follows that $P(\theta \in C_{\mathbf{w}}) = P((\theta_1'\theta_1)^2 = q^2(\theta_2'\theta_2) = r^4) = 0$ and $P(\theta \notin C_{\mathbf{w}}) = 1$, and further, $(\chi_k^2 + \chi_s^2)$ is independent of (χ_k^2/χ_s^2).

Finally, $P(S(C_{\mathbf{w}})) = P((r|\theta_1| + \frac{r^2}{q}|\theta_2|)^2 \leq \frac{c^2}{\nu}p\ \chi_\nu^2(r^2 + \frac{r^4}{q^2}))$

$$= P((r\chi_k + \frac{r^2}{q}\chi_s)^2 \leq \frac{c^2}{\nu}p\ \chi_\nu^2(r^2 + \frac{r^4}{q^2}))$$

$$= P((\chi_k + \frac{r}{q}\chi_s)^2 \leq \frac{c^2}{\nu}p\ \chi_\nu^2(1 + \frac{r^2}{q^2})).$$ .QED.

Proof of Lemma 1 :

$P((\chi_k + \frac{r}{q}\chi_s)^2 \leq \frac{c^2}{\nu} p\ \chi^2_\nu (1 + \frac{r^2}{q^2}),\ \chi^2_k + \chi^2_s < \frac{c^2}{\nu} p\ \chi^2_\nu)$

$= P((\chi^2_k + 2\frac{r}{q}\chi_k\chi_s + \frac{r^2}{q^2}\chi^2_s) \leq \frac{c^2}{\nu} p\chi^2_\nu(1 + \frac{r^2}{q^2}),\ (\chi^2_k + \chi^2_s)(1 + \frac{r^2}{q^2}) < \frac{c^2}{\nu} p\chi^2_\nu(1 + \frac{r^2}{q^2}))$

$= P((\chi^2_k + \chi^2_s)(1 + \frac{r^2}{q^2}) < (\frac{c^2}{\nu} p\ \chi^2_\nu)(1 + \frac{r^2}{q^2}))$

$= P(\chi^2_p \leq \frac{c^2}{\nu} p\ \chi^2_\nu)$

$= P(F_{p,\nu} \leq c^2).$ .QED.

Proof of Lemma 2:

Let $U = \chi^2_k + \chi^2_s \sim \chi^2_p$, and $V = \chi_k/\chi_s$; note U and V are independent. Then, $(\chi_k + \frac{r}{q}\chi_s)^2 = \chi^2_s(\frac{\chi_k}{\chi_s} + \frac{r}{q})^2 = \chi^2_s(V + \frac{r}{q})^2 = (\frac{\chi^2_s}{\chi^2_k + \chi^2_s})(\chi^2_k + \chi^2_s)(V + \frac{r}{q})^2 = \frac{U(V + \frac{r}{q})^2}{V^2 + 1}$.

Substituting into (3.1), then

$P(\frac{U(V+\frac{r}{q})^2}{V^2+1} \leq \frac{c^2}{\nu} p\ \chi^2_\nu(1 + \frac{r^2}{q^2}),\ \chi^2_k + \chi^2_s \geq \frac{c^2}{\nu} p\ \chi^2_\nu)$

$= P(\frac{(V+\frac{r}{q})^2}{V^2+1} - (1 + \frac{r^2}{q^2})(c^2/F_{p,\nu}) \leq 0,\ F_{p,\nu} \geq c^2)$

$= P((1 - \frac{c^2(1+r^2/q^2)}{F_{p,\nu}})V^2 + 2(r/q)V + (\frac{r^2}{q^2} - \frac{c^2(1+r^2/q^2)}{F_{p,\nu}}) \leq 0,\ F_{p,\nu} \geq c^2)$

$= P(F_{p,\nu} \geq c^2(1 + (r^2/q^2)),\ \ F_{p,\nu} \geq c^2,\quad (V - \lambda_1)(V - \lambda_2) \leq 0)$ (3.2)

$+P(F_{p,\nu} < c^2(1 + (r^2/q^2)),\ \ F_{p,\nu} \geq c^2,\quad (V - \lambda_1)(V - \lambda_2) > 0)$ (3.3)

where $\lambda_1 = \lambda_1(F_{p,\nu}) = (-\frac{r}{q}F_{p,\nu} - (1 + \frac{r^2}{q^2})\sqrt{F_{p,\ \nu}c^2 - c^4})/(F_{p,\nu} - c^2(1 + \frac{r^2}{q^2}))$ and $\lambda_2 = \lambda_2(F_{p,\nu}) = (-\frac{r}{q}F_{p,\nu} + (1 + \frac{r^2}{q^2})\sqrt{F_{p,\ \nu}c^2 - c^4})/(F_{p,\nu} - c^2(1 + \frac{r^2}{q^2}))$.

Under the condition $F_{p,\nu} < c^2(1 + (r^2/q^2))$, λ_1 is always greater than λ_2. so, (3.3) is equivalent to

$P(F_{p,\nu} < c^2(1 + (r^2/q^2)),\ F_{p,\nu} \geq c^2,\ (V \leq \lambda_2 \text{ or } V \geq \lambda_1),\ \lambda_2 < \lambda_1)$

$= P(F_{p,\nu} < c^2(1 + (r^2/q^2)),\ F_{p,\nu} \geq c^2,\ V \leq \lambda_2,\ \lambda_2 < 0 < \lambda_1)$ (3.3.1)

$+P(F_{p,\nu} < c^2(1 + (r^2/q^2)),\ F_{p,\nu} \geq c^2,\ V \leq \lambda_2\ ,\ 0 < \lambda_2 < \lambda_1)$ (3.3.2)

$+P(F_{p,\nu} < c^2(1 + (r^2/q^2)),\ F_{p,\nu} \geq c^2,\ V > \lambda_1,\ 0 < \lambda_2 < \lambda_1)$ (3.3.3)

$$+P(F_{p,\nu} < c^2(1+(r^2/q^2)),\ F_{p,\nu} \geq c^2,\ V > \lambda_1,\ \lambda_2 < 0 < \lambda_1) \tag{3.3.4}$$

Since $V > 0$, $P(F_{p,\nu} < c^2(1+(r^2/q^2)),\ F_{p,\nu} \geq c^2,\ V \leq \lambda_2,\ \lambda_2 < 0 < \lambda_1) = 0$. This leaves expressions (3.3.2), (3.3.3) and (3.3.4). Also, since $F_{p,\nu} < c^2(1+(r^2/q^2))$, the denominator of λ_2 is negative. So, for $\lambda_2 < 0$, the numerator must be positive. That is, $-\frac{r}{q}F_{p,\nu} + (1+\frac{r^2}{q^2})\sqrt{F_{p,\ \nu}c^2 - c^4} > 0 => (\frac{r^2}{q^2}F_{p,\nu} - c^2(1+\frac{r^2}{q^2}))(F_{p,\nu} - c^2(1+\frac{r^2}{q^2})) < 0$. The solution will be one of the following two cases:

a1. $\frac{r^2}{q^2} \geq 1 => c^2(1+(q^2/r^2)) \leq F_{p,\nu} \leq c^2(1+(r^2/q^2))$

a2. $\frac{r^2}{q^2} < 1 => c^2(1+(r^2/q^2)) \leq F_{p,\nu} \leq c^2(1+(q^2/r^2))$

Similarly, when $\lambda_2 > 0$, the two cases are:

b1. $\frac{r^2}{q^2} \geq 1 => (F_{p,\nu} < c^2(1+(q^2/r^2))$ or $(F_{p,\nu} > c^2(1+(r^2/q^2))$

b2. $\frac{r^2}{q^2} < 1 => (F_{p,\nu} < c^2(1+(r^2/q^2))$ or $(F_{p,\nu} > c^2(1+(q^2/r^2))$

So, (3.3)

$= P(F_{p,\nu} < c^2(1+(r^2/q^2)),\ F_{p,\nu} \geq c^2,\ V \leq \lambda_2,\ (F_{p,\nu} < c^2(1+(q^2/r^2))$ or

$(F_{p,\nu} > c^2(1+(r^2/q^2)),\ r^2/q^2 \geq 1)$

$+P(F_{p,\nu} < c^2(1+(r^2/q^2)),\ F_{p,\nu} \geq c^2,\ V \leq \lambda_2,\ (F_{p,\nu} < c^2(1+(r^2/q^2))$ or

$(F_{p,\nu} > c^2(1+(q^2/r^2)),\ r^2/q^2 < 1)$

$+P(F_{p,\nu} < c^2(1+(r^2/q^2)),\ F_{p,\nu} \geq c^2,\ V \geq \lambda_1 > \lambda_2 > 0,\ (F_{p,\nu} < c^2(1+$

$(q^2/r^2))$ or $(F_{p,\nu} > c^2(1+(r^2/q^2)),\ r^2/q^2 \geq 1)$

$+P(F_{p,\nu} < c^2(1+(r^2/q^2)),\ F_{p,\nu} \geq c^2,\ V \geq \lambda_1 > \lambda_2 > 0,\ (F_{p,\nu} < c^2(1+$

$(r^2/q^2))$ or $(F_{p,\nu} > c^2(1+(q^2/r^2)),\ r^2/q^2 < 1)$

$+P(F_{p,\nu} < c^2(1+(r^2/q^2)),\ F_{p,\nu} \geq c^2,\ V > \lambda_1 > 0 > \lambda_2,\ c^2(1+(q^2/r^2)) \leq$

$F_{p,\nu} \leq c^2(1+(r^2/q^2)),\ r^2/q^2 \geq 1)$

$+P(F_{p,\nu} < c^2(1+(r^2/q^2)),\ F_{p,\nu} \geq c^2,\ V > \lambda_1 > 0 > \lambda_2,\ c^2(1+(r^2/q^2)) \leq$

$F_{p,\nu} \leq c^2(1+(q^2/r^2)),\ r^2/q^2 < 1)$

$= P(0 \le V \le \lambda_2,\, c^2 \le F_{p,\nu} \le c^2(1 + (q^2/r^2)),\, r^2/q^2 \ge 1)$

$+P(0 \le V \le \lambda_2,\, c^2 \le F_{p,\nu} \le c^2(1 + (r^2/q^2)),\, r^2/q^2 < 1)$

$+P(V \ge \lambda_1,\, c^2 \le F_{p,\nu} \le c^2(1 + (q^2/r^2)),\, r^2/q^2 \ge 1)$

$+P(V \ge \lambda_1,\, c^2 \le F_{p,\nu} \le c^2(1 + (r^2/q^2)),\, r^2/q^2 < 1)$

$+P(V \ge \lambda_1,\, c^2(1 + (q^2/r^2) \le F_{p,\nu} \le c^2(1 + (r^2/q^2)),\, r^2/q^2 \ge 1)$

$= P(0 \le V \le \lambda_2,\, c^2 \le F_{p,\nu} \le c^2(1 + (q^2/r^2)),\, r^2/q^2 \ge 1)$

$+P(0 \le V \le \lambda_2,\, c^2 \le F_{p,\nu} \le c^2(1 + (r^2/q^2)),\, r^2/q^2 < 1)$

$+P(V \ge \lambda_1,\, c^2 \le F_{p,\nu} \le c^2(1 + (r^2/q^2)),\, r^2/q^2 \ge 1)$

$+P(V \ge \lambda_1,\, c^2 \le F_{p,\nu} \le c^2(1 + (r^2/q^2)),\, r^2/q^2 < 1)$

In a similar way, it can be shown that (3.2) is equal to

$P(0 \le V \le \lambda_2,\, c^2(1 + (r^2/q^2)) \le F_{p,\nu} \le c^2(1 + (q^2/r^2)),\, r^2/q^2 < 1)$

Thus the lemma is established. .QED.

Proof of Theorem 2 :

$P\{(\chi_k + \frac{r}{q}\chi_s)^2 \le \frac{c^2}{\nu} p\; \chi^2_\nu(1 + (r^2/q^2)\}$

$= P\{(\chi_k + \frac{r}{q}\chi_s)^2 \le \frac{c^2}{\nu} p\; \chi^2_\nu(1 + \frac{r^2}{q^2}),\, \chi^2_k + \chi^2_s < \frac{c^2}{\nu} p\; \chi^2_\nu\}$

$+P\{(\chi_k + \frac{r}{q}\chi_s)^2 \le \frac{c^2}{\nu} p\; \chi^2_\nu(1 + \frac{r^2}{q^2}),\, \chi^2_k + \chi^2_s \ge \frac{c^2}{\nu} p\; \chi^2_\nu\}$

The result follows from lemma 1, lemma 2 and $\frac{s}{k}V^2 = F_{k,s}$. .QED.

Proof of Theorem 3:

Let $S(C_{\mathbf{u}}) = \{\beta_{\mathbf{u}} : (\mathbf{u}'\hat{\beta}_{\mathbf{u}} - \mathbf{u}'\beta_{\mathbf{u}})^2 \le c^2 p s^2 \mathbf{u}'V\mathbf{u},\, \forall \mathbf{u} \in C_{\mathbf{u}}\}$,

where $C_{\mathbf{u}} = \{\mathbf{u} : (\mathbf{u}_1'\mathbf{u}_1)^2 = (\mathbf{u}_2'\mathbf{u}_2) = r_0^2\}$. Let $\mathbf{w} = V^{1/2}\mathbf{u}$ and $\beta_{\mathbf{w}} = V^{-1/2}\beta_{\mathbf{u}}$. Then, $\mathbf{u}'\beta_{\mathbf{u}} = \mathbf{w}'\beta_{\mathbf{w}}$ and $\mathbf{u}'V\mathbf{u} = \mathbf{w}'\mathbf{w}$.

The corresponding $C_{\mathbf{w}} = \{\mathbf{w} : V^{-1/2}\mathbf{w} \in C_{\mathbf{u}}\}$

$= \{\mathbf{w} = (\mathbf{w}_1',\, \mathbf{w}_2')' : (\frac{1}{\sqrt{a}}\mathbf{w}_1',\, \frac{1}{\sqrt{b}}\mathbf{w}_2')' \in C_{\mathbf{u}}\}$

$= \{\mathbf{w} : \frac{1}{a^2}(\mathbf{w}_1'\mathbf{w}_1)^2 = \frac{1}{b}(\mathbf{w}_2'\mathbf{w}_2) = r_0^2\}$

$= \{\mathbf{w} : (\mathbf{w}_1'\mathbf{w}_1)^2 = \frac{a^2}{b}(\mathbf{w}_2'\mathbf{w}_2) = a^2 r_0^2\}.$

$P(S(C_{\mathbf{u}})) = P(\frac{(\mathbf{u}'\hat{\beta}_{\mathbf{u}} - \mathbf{u}'\beta_{\mathbf{u}})^2}{ps^2\mathbf{u}'V\mathbf{u}} \leq c^2, \forall \mathbf{u} \in C_{\mathbf{u}}\}$

$= P(\frac{(\mathbf{w}'\hat{\beta}_{\mathbf{w}} - \mathbf{w}'\beta_{\mathbf{w}})^2}{ps^2\mathbf{w}'\mathbf{w}} \leq c^2, \forall \mathbf{w} \in C_{\mathbf{w}}\}$

where $C_{\mathbf{w}} = \{\mathbf{w} : (\mathbf{w}_1'\mathbf{w}_1)^2 = q^2(\mathbf{w}_2'\mathbf{w}_2) = r^4\}$ with $q^2 = a^2/b$ and $r^4 = a^2 r_0^2$. .QED.

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