Computing Probability Integrals of 
a Bivariate Normal Distribution

by

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ABSTRACT

The bivariate normal integral has been computed and tabulated in several sources in the past 40 years. The objective of this paper is two-fold. First, matrix algebra and geometry is used to reduce the complexity of evaluating the normal double integral thru a transformation to a single integral of the cumulative univariate standardized normal distribution. As a result, the accuracy for the volume under the bivariate normal density is at least through 7 decimals.

Secondly, the objective is to present a very user-friendly software that computes the volume over both elliptical and rectangular regions. The user may specify the level of accuracy by inputting the number of subintervals for the Simpson's Rule.

Key Words: Bivariate Normal Integral, Matrix Transformation, Computer Software.
I. INTRODUCTION

If the random vector, $X$, has a $p$-variate normal distribution, then

$$X'\Sigma^{-1}X = X' \left( \sum_{i=1}^{p} \left( \frac{1}{\lambda_i} \right) e_i e_i' \right) X$$

$$= \sum_{i=1}^{p} \left( \frac{1}{\lambda_i} \right) (X'e_i)^2 \geq 0,$$  \hspace{1cm} (1)

where $\Sigma$ is the covariance matrix, $\lambda_i$'s, $i=1,2,\ldots,p$ are the eigenvalues of $\Sigma$ with corresponding eigenvectors $e_i$.

Since the quadratic form $X'\Sigma X$ is positive definite (i.e., $X'\Sigma X > 0$ for any vector $X$ and is zero only when $X$ is the zero vector), $\lambda_i$'s > 0 and the contours of constant density for a $p$-dimensional normal distribution are ellipsoids defined by:

$$(X-\mu)'\Sigma^{-1}(X-\mu) = c^2.$$  \hspace{1cm} (2)

These ellipsoids are centered at the mean vector $\mu$ and have axes $\pm c\sqrt{\lambda_i}e_i$, where $\Sigma e_i = \lambda_i e_i$, $i=1,2,\ldots,p$. The major axis of the ellipsoid is along the eigenvector with maximum $\lambda$ [1].

Equation (2) suggests a translation of axes to $\mu'=(\mu_1, \mu_2, \ldots, \mu_p)$ followed by a rotation of the translated axes to the axes of the ellipsoid. After rotation, the covariance matrix for the new axes will be
The above transformation will reduce the complexity of evaluating the multivariate normal integral over a specified region.

The objective of this paper is to develop a method that uses a personal computer for evaluating the bivariate normal integral, i.e., the case of \( p=2 \). The integration is carried out over the area of an ellipse and also over a rectangle.

A \( p \)-dimensional normal density function for the random vector \( X = [X_1, X_2, \ldots, X_p]' \) is given by

\[
\begin{pmatrix}
\lambda_1 \\
\lambda_2 & 0 \\
\vdots & \ddots & \ddots \\
0 & \ddots & \ddots & 0 \\
\lambda_p
\end{pmatrix}
\]

(3)

A \( p \)-dimensional normal density function for the random vector \( X=[X_1,X_2,\ldots,X_p]' \) is given by

\[

f(x) = \frac{1}{(2\pi)^{p/2} \sqrt{\text{det}(\Sigma)}} e^{-\frac{1}{2}(x-\mu)'\Sigma^{-1}(x-\mu)},
\]

(4)

where \(-\infty < X_i < \infty, i=1,2,\ldots,p., \) and \( X \) is said to have a \( N_p(\mu,\Sigma) \) distribution.

For the case of \( N_2(\mu,\Sigma) \), the bivariate normal density function is
\[ \mathcal{L}(x_1, x_2) = \frac{1}{2\pi\sqrt{\sigma_{11}\sigma_{22}(1-\rho^2)}} \exp \left\{ -\frac{1}{2(1-\rho^2)} \left[ \left( \frac{x_1 - \mu_1}{\sqrt{\sigma_{11}}} \right)^2 - 2\rho \left( \frac{x_1 - \mu_1}{\sqrt{\sigma_{11}}} \right) \left( \frac{x_2 - \mu_2}{\sqrt{\sigma_{22}}} \right) + \left( \frac{x_2 - \mu_2}{\sqrt{\sigma_{22}}} \right)^2 \right] \right\} \] 

(5)

where \( \Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} \), and \( \sigma_{12} = \sigma_{21} \) is the covariance between \( X_1 \) and \( X_2 \).
II. INTEGRATION OVER AN ELLIPSE

In multivariate calculus, we have the result

$$\iint_{\Lambda} \xi(x, y) \, dx \, dy = \iint_{\Lambda'} F(u_1, u_2) \left| \frac{\partial (x, y)}{\partial (u_1, u_2)} \right| \, du_1 \, du_2, \quad (6)$$

where $F(u_1, u_2) = f[x(u_1, u_2), y(u_1, u_2)]$ transforms $\Lambda$ into $\Lambda'$, provided that $\frac{\partial (x, y)}{\partial (u_1, u_2)}$ is continuous and nonzero in $\Lambda'$.

Thus, any double integral of the form $\iint_{\Lambda} \xi(x, y) \, dx \, dy$,

where $\Lambda$ is the elliptical region

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

can be transformed (thru $x = a \cos \theta$, $y = b \sin \theta$) as follows:

$$\iint_{\Lambda} \xi(x, y) \, dx \, dy = ab \int_{0}^{2\pi} \int_{0}^{1} \xi(\arccos u, b \sin \theta) \, r \, dr \, d\theta. \quad (7)$$

We can use equation (7) first to reduce the bivariate normal integral to a univariate integral and then compute it accurately.

The use of spectral decomposition results in the following elliptical contours:
\[(X-\mu)/\Sigma^{-1}(X-\mu)\]
\[
= [x_1 - \mu_1, x_2 - \mu_2] \Sigma^{-1} \begin{bmatrix} x_1 - \mu_1 \\ x_2 - \mu_2 \end{bmatrix}
\]
\[
= [x_1 - \mu_1, x_2 - \mu_2] \sum_{i=1}^{2} \lambda_i^{-1} \theta_i \phi_i \begin{bmatrix} x_1 - \mu_1 \\ x_2 - \mu_2 \end{bmatrix}
\]
\[
= \sum_{i=1}^{2} \lambda_i^{-1} [(X-\mu)'\theta_i]^2
\]
\[
= \frac{1}{\lambda_1} [(X-\mu)'\phi_1]^2 + \frac{1}{\lambda_2} [(X-\mu)'\phi_2]^2
\]
\[
= \frac{\xi_1^2}{\lambda_1} + \frac{\xi_2^2}{\lambda_2} = \chi^2_g(2),
\]

where \(\chi^2_g(2)\) represents the \(\alpha \times 100\) percentage point of a Chi-square random variable with 2 degrees of freedom (df).

Further, due to the fact that \(\bar{x}_i = (X-\mu)'e_i, i=1,2,...,p\) is \(N_1(0, \lambda_i)\), then \(\xi_i/\sqrt{\lambda_i}\) is \(N_1(0,1)\) and the sum in (8) has a \(\chi^2\) distribution with 2 df. Equation (8) results in the elliptical contours:
\[
\frac{\xi_1^2}{\chi^2_g(2) \lambda_1} + \frac{\xi_2^2}{\chi^2_g(2) \lambda_2} = 1,
\]

where the length of semimajor axis of the ellipse is \(\sqrt{\lambda_1 \chi^2_g(2)}\) and that of semiminor axis is \(\sqrt{\lambda_2 \chi^2_g(2)}\).

After the origin is translated to \([\mu_1, \mu_2]\) , the center of
elliptical region is now\( (0,0) \), and a rotation of axes thru the angle \( \Psi = \arctan((\lambda_1-\sigma_{11})/\sigma_{12}) \) reduces the density function to

\[
\tilde{f}(\tilde{x}_1, \tilde{x}_2) = \left( \frac{1}{2\pi \sqrt{\lambda_1 \lambda_2}} \right) \exp \left\{ -\frac{1}{2} \left[ \left( \frac{\tilde{x}_1}{\sqrt{\lambda_1}} \right)^2 + \left( \frac{\tilde{x}_2}{\sqrt{\lambda_2}} \right)^2 \right] \right\}
\]

where \( \tilde{x}_1 = x_1 \cos \Psi + x_2 \sin \Psi \) and \( \tilde{x}_2 = -x_1 \sin \Psi + x_2 \cos \Psi \).

Combining equations (7) and (10) gives the probability over an elliptical region, \( P_E \), as shown below.

\[
P_E = \int \int \frac{1}{2\pi \sqrt{\lambda_1 \lambda_2}} \times \exp \left\{ -\frac{1}{2} \left[ \left( \frac{\tilde{x}_1}{\sqrt{\lambda_1}} \right)^2 + \left( \frac{\tilde{x}_2}{\sqrt{\lambda_2}} \right)^2 \right] \right\} \, d\tilde{x}_1 \, d\tilde{x}_2
\]

\[
= ab \int_0^{2\pi} \int_0^1 \frac{1}{2\pi \sqrt{\lambda_1 \lambda_2}} \times \exp \left\{ -\frac{1}{2} \left[ \left( \frac{a \cos \theta}{\sqrt{\lambda_1}} \right)^2 + \left( \frac{b \sin \theta}{\sqrt{\lambda_2}} \right)^2 \right] \right\} \, r \, dr \, d\theta
\]

\[
= \frac{ab}{2\pi \sqrt{\lambda_1 \lambda_2}} \int_0^{2\pi} \int_0^1 \exp \left\{ -\frac{1}{2} \left[ \left( \frac{a \cos \theta}{\sqrt{\lambda_1}} \right)^2 + \left( \frac{b \sin \theta}{\sqrt{\lambda_2}} \right)^2 \right] \right\} \, r \, dr \, d\theta
\]

where \( a = \sqrt{\lambda_1 \lambda_2} \(2) \) and \( b = \sqrt{\lambda_3 \lambda_3} \(2) \).

Letting \( y = r^2/2 \) results in
Therefore, given the mean vector $\mu$, the covariance matrix $\Sigma$ and the length of semimajor axis $a$, the value of $a^2/\lambda_1$ can be substituted for $\chi^2_m(2)$ in (14) to compute $P_E$. This results in the following probability over an elliptical region.

$$P_E = 1 - e^{-\frac{a^2}{\lambda_1}}.$$
III. INTEGRATION OVER A RECTANGLE

The bivariate normal probability integral over a rectangular region has been evaluated by many approximation methods [2], [3], [4], [6] and [11]. However, they are either laborious (time-consuming) or not very accurate. This section will derive a method to accurately approximate the bivariate normal integral over a rectangular region.

The bivariate normal integral over a rectangular region \(a \leq x_1 \leq b_1\) and \(a_2 \leq x_2 \leq b_2\) is given by

\[
P_{X} = \int_{a_2}^{b_2} \int_{a_1}^{b_1} f(X_1, X_2) \, dX_1 \, dX_2 = \int_{a_2}^{b_2} \int_{a_1}^{b_1} \frac{1}{2\pi \sigma_{11} \sigma_{22} (1-\rho^2)} \exp \left\{ -\frac{1}{2(1-\rho^2)} \left[ \frac{(X_1-\mu_1)^2}{\sigma_{11}} - 2\rho \frac{(X_1-\mu_1)(X_2-\mu_2)}{\sigma_{11}\sigma_{22}} + \frac{(X_2-\mu_2)^2}{\sigma_{22}} \right] \right\} \, dX_1 \, dX_2
\]  

Substituting \(Z_1 = \frac{X_1-\mu_1}{\sigma_{11}}\), \(Z_2 = \frac{X_2-\mu_2}{\sigma_{22}}\) into (15) yields

\[
\int_{a_2'}^{b_2'} \int_{a_1'}^{b_1'} \frac{1}{2\pi \sqrt{1-\rho^2}} \exp \left\{ -\frac{1}{2(1-\rho^2)} \left[ Z_1^2 - 2\rho Z_1 Z_2 + Z_2^2 \right] \right\} \, dZ_1 \, dZ_2
\]  

where \(a_1' = \frac{a_1-\mu_1}{\sigma_{11}}\), \(b_1' = \frac{b_1-\mu_1}{\sigma_{11}}\), \(a_2' = \frac{a_2-\mu_2}{\sigma_{22}}\), \(b_2' = \frac{b_2-\mu_2}{\sigma_{22}}\), and
the covariance matrix for the bivariate vector \( Z = [z_1 z_2]' \) is

\[
\Sigma_z = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}.
\]

(17)

When the \( Z_1-Z_2 \) axes are rotated, the rotation angle will always be 45° (see Fig. A1) because \( Z_1 \) and \( Z_2 \) have equal unit variances.

We first rotate the \( Z_1-Z_2 \) axis such that \( Z' \Sigma_z^{-1} Z \) transforms into a quadratic form with no interaction term. The use of spectral decomposition leads to

\[
Z' \Sigma_z^{-1} Z = Z' \left( \sum_{i=1}^{2} \left( \frac{1}{\lambda_i} \right) e_i e_i' \right) Z
\]

\[
= \sum_{i=1}^{2} \left( \frac{1}{\lambda_i} \right) (Z' e_i)^2
\]

\[
= \frac{\nu_1^2}{\lambda_1} + \frac{\nu_2^2}{\lambda_2},
\]

(18)

where \((\lambda_i, e_i)\) are now eigenvalue-eigenvector pairs of \( \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \), so that

\[
e_1 = \begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \\ \sqrt{2} \end{bmatrix}, \quad e_2 = \begin{bmatrix} -1 \\ \sqrt{2} \\ -1 \\ \sqrt{2} \end{bmatrix}, \quad \Psi = 45°, \text{ and}
\]

\(\lambda_1 = 1 + \rho\) and \(\lambda_2 = 1 - \rho\). This will lead to
\[ w_1 = e_1 Z \]

\[ = \begin{bmatrix} 1 \\ \frac{1}{\sqrt{2}} \end{bmatrix} Z_1 \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 1 \end{bmatrix} Z_2 \]

\[ = \frac{1}{\sqrt{2}} (Z_1 + Z_2) \]

and

\[ w_2 = e_2 Z \]

\[ = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} Z_1 \begin{bmatrix} 1 \\ \frac{1}{\sqrt{2}} \end{bmatrix} Z_2 \]

\[ = \frac{1}{\sqrt{2}} (-Z_1 + Z_2) \].

From Figure A_2, we obtain

\[ a_1''' = \frac{1}{\sqrt{2}} (b_1' + b_2') \]

\[ b_1''' = \frac{1}{\sqrt{2}} (-b_1' + b_2') \]

\[ a_2''' = \frac{1}{\sqrt{2}} (a_1' + b_2') \]

\[ b_2''' = \frac{1}{\sqrt{2}} (-a_1' + b_2') \]

\[ a_3''' = \frac{1}{\sqrt{2}} (a_1' + a_2') \]

\[ b_3''' = \frac{1}{\sqrt{2}} (-a_1' + a_2') \]

\[ a_4''' = \frac{1}{\sqrt{2}} (b_1' + a_2') \]

\[ b_4''' = \frac{1}{\sqrt{2}} (-b_1' + a_2') \]

and the constants:
After rotation, the covariance matrix in the $W_1 - W_2$ coordinate system is:

$$
\Sigma_{\psi} = \begin{bmatrix}
1 + \rho & 0 \\
0 & 1 - \rho
\end{bmatrix}
$$

and thus, equation (16) becomes:

$$
P_R = \int \int \frac{1}{2\pi \sqrt{1-\rho^2}} e^{-\frac{1}{2} \left( \frac{\psi_1^2}{\kappa_1^2} + \frac{\psi_2^2}{\kappa_2^2} \right)} d\psi_1 d\psi_2
$$

where $A'$ is the rectangular region in Figure A₂. Note that $A'$ is divided into three sections (1), (2) and (3) as shown in Figure A₂. Hence, the integral in equation (24) will equal to:
\[
\begin{align*}
P_R &= \int_{s_N} \int_{s_{N-1}} \int_{s_{N-2}} \cdots \int_{s_1} \int_{s_0} \frac{1}{2\pi \sqrt{1-\rho^2}} \exp \left( -\frac{1}{3} \left( \frac{\omega_1^2 + \omega_2^2}{\lambda_1^2 + \lambda_2^2} \right) \right) \, d\omega_1 \, d\omega_2 \\
+ \int_{s_N} \int_{s_{N-1}} \int_{s_{N-2}} \cdots \int_{s_1} \int_{s_0} \frac{1}{2\pi \sqrt{1-\rho^2}} \exp \left( -\frac{1}{3} \left( \frac{\omega_1^2 + \omega_2^2}{\lambda_1^2 + \lambda_2^2} \right) \right) \, d\omega_1 \, d\omega_2 \\
+ \int_{s_N} \int_{s_{N-1}} \int_{s_{N-2}} \cdots \int_{s_1} \int_{s_0} \frac{1}{2\pi \sqrt{1-\rho^2}} \exp \left( -\frac{1}{3} \left( \frac{\omega_1^2 + \omega_2^2}{\lambda_1^2 + \lambda_2^2} \right) \right) \, d\omega_1 \, d\omega_2 \\
+ \int_{s_N} \int_{s_{N-1}} \int_{s_{N-2}} \cdots \int_{s_1} \int_{s_0} \frac{1}{2\pi \sqrt{1-\rho^2}} \exp \left( -\frac{1}{3} \left( \frac{\omega_1^2 + \omega_2^2}{\lambda_1^2 + \lambda_2^2} \right) \right) \, d\omega_1 \, d\omega_2 \\
&= \int_{s_N} \int_{s_{N-1}} \int_{s_{N-2}} \cdots \int_{s_1} \int_{s_0} \frac{1}{2\pi \sqrt{1-\rho^2}} \exp \left( -\frac{1}{3} \left( \frac{\omega_1^2 + \omega_2^2}{\lambda_1^2 + \lambda_2^2} \right) \right) \, d\omega_1 \, d\omega_2
\end{align*}
\]

(25)

Now, let \( y_1 = \frac{\omega_1}{\sqrt{\lambda_1}} \) and \( y_2 = \frac{\omega_2}{\sqrt{\lambda_2}} \) in (25).

\[
\begin{align*}
P_R &= \int_{s_N} \int_{s_{N-1}} \int_{s_{N-2}} \cdots \int_{s_1} \int_{s_0} \frac{1}{2\pi \sqrt{1-\rho^2}} \exp \left( -\frac{1}{3} \left( y_1^2 + y_2^2 \right) \right) \sqrt{\frac{1}{\lambda_1 \lambda_2}} \, dy_1 \, dy_2 \\
+ \int_{s_N} \int_{s_{N-1}} \int_{s_{N-2}} \cdots \int_{s_1} \int_{s_0} \frac{1}{2\pi \sqrt{1-\rho^2}} \exp \left( -\frac{1}{3} \left( y_1^2 + y_2^2 \right) \right) \sqrt{\frac{1}{\lambda_1 \lambda_2}} \, dy_1 \, dy_2 \\
+ \int_{s_N} \int_{s_{N-1}} \int_{s_{N-2}} \cdots \int_{s_1} \int_{s_0} \frac{1}{2\pi \sqrt{1-\rho^2}} \exp \left( -\frac{1}{3} \left( y_1^2 + y_2^2 \right) \right) \sqrt{\frac{1}{\lambda_1 \lambda_2}} \, dy_1 \, dy_2 \\
+ \int_{s_N} \int_{s_{N-1}} \int_{s_{N-2}} \cdots \int_{s_1} \int_{s_0} \frac{1}{2\pi \sqrt{1-\rho^2}} \exp \left( -\frac{1}{3} \left( y_1^2 + y_2^2 \right) \right) \sqrt{\frac{1}{\lambda_1 \lambda_2}} \, dy_1 \, dy_2 \\
&= \int_{s_N} \int_{s_{N-1}} \int_{s_{N-2}} \cdots \int_{s_1} \int_{s_0} \frac{1}{2\pi \sqrt{1-\rho^2}} \exp \left( -\frac{1}{3} \left( y_1^2 + y_2^2 \right) \right) \sqrt{\frac{1}{\lambda_1 \lambda_2}} \, dy_1 \, dy_2
\end{align*}
\]

(26)
\[
\begin{align*}
\int \frac{a}{\sqrt{\lambda}} \int \frac{a}{\sqrt{\lambda}} \frac{\nu_k + c_k}{\sqrt{\lambda}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} dy_1 dy_2 \\
+ \int \frac{a}{\sqrt{\lambda}} \int \frac{a}{\sqrt{\lambda}} \frac{-\nu_k + c_k}{\sqrt{\lambda}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} dy_1 dy_2 \\
+ \int \frac{a}{\sqrt{\lambda}} \int \frac{a}{\sqrt{\lambda}} \frac{\nu_k - c_k}{\sqrt{\lambda}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} dy_1 dy_2 \\
+ \int \frac{a}{\sqrt{\lambda}} \int \frac{a}{\sqrt{\lambda}} \frac{-\nu_k - c_k}{\sqrt{\lambda}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} dy_1 dy_2 
\end{align*}
\]
where $\Phi$ is the univariate standard normal cumulative distribution function. Therefore, we have efficiently reduced the double normal integral to a single integral of $\Phi$ (which can be computed using several approximation methods \[7\] and specifically Moran's approximation \[8\]). As a result, we can use Simpson's rule or other approximation methods to accurately approximate the single integral in (28).

The methods for different situations like Figures B, C, and D are similar to that of Figure A, and hence will not be repeated.
IV. SUMMARY AND CONCLUSIONS

The major goal of this paper was to develop a method to accurately evaluate a bivariate normal integral by a personal computer.

In section II, the integration over an elliptical region led to the following simple formula:

\[ P_2 = 1 - e^{-\frac{s^2}{2\lambda_1}}. \]

where \( \lambda_1 \) is the maximum eigenvalue of the covariance matrix \( \Sigma \), and \( a \) is the semi-major axis of the ellipse.

Integration over a rectangle was developed in section III. Its accuracy depends on the approximation method that evaluates the cumulative standard normal and the number of subintervals that is used in Simpson's rule. It was determined that the very accurate Moran's approximation to the cumulative normal [8] gives up to 9 decimal place accuracy. The accuracy in Simpson's rule depends on the number of subintervals into which the range of the single integral in (28) is divided. The results for finite intervals were compared for 50, 100, 200, 300, and 1000 subintervals. Since 1000 subintervals gave the same accuracy as those of 200 intervals thru 6 decimals and because of excessive computing time, only 200 subintervals are recommended. Consequently, the integration results over a rectangular region should be more accurate than those that had been reported before.
Table 1 compares probability integral for a bivariate normal distribution from (28) at $\rho=0.5$ to those generated by two other sources. The top entry is by Goodman[9], and calculated from the National Bureau of Standard's tables [10], and the second entry is generated by iterative proportional fitting [4]. The bottom entry is the values computed by (28), where Probability=entry/10000.

For example, using equation (28), the entry in the second column and ninth row of Table 1 gives the value of:

$$
\int_{x_1=1.5}^{x_1=1.0} \int_{x_2=0.5}^{x_2=1.0} N_2 \left( \begin{bmatrix} 0 \\ 0 \\ 1 \\ .50 \\ .50 \\ 1 \end{bmatrix} \right) dx_1 dx_2
$$

which is .009904, while the entry in the last column and 3rd row of Table 1 gives the value of

$$
\int_{x_1=1.5}^{x_1=1.0} \int_{x_2=0.5}^{x_2=1.0} N_2 \left( \begin{bmatrix} 0 \\ 0 \\ 1 \\ .50 \\ .50 \\ 1 \end{bmatrix} \right) dx_1 dx_2
$$

which is .000669.

The program was developed utilizing the C programming language. The software, which is very user friendly, is available on request on a 3.5" diskette. The main executable program in BIV.EXE. The user is asked to input the values of $\mu_1$, $\mu_2$, $\sigma_{11}$, $\sigma_{12}$ and $\sigma_{22}$. Then the software inquires if the integration is carried out over an ellipse (e) or a rectangle
(r). If the user responds (r), then the following information will be asked for: The limits of $x_1$, the limits of $x_2$ and the number of subinterval for the use of Simpson's rule. If the user inputs (e), then the software asks only for the value of long semiaxis $a$. Although Table 1 provides between 3 to 5 decimal accuracy, the software outputs 8 decimals with accuracy depending on the number of subintervals the user inputs for the Simpson's rule.
\( p = 0.50 \)

**TABLE 1**

<table>
<thead>
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<th>( x_i = 0.0 )</th>
<th>( x_i = 0.5 )</th>
<th>( x_i = 1.0 )</th>
<th>( x_i = 1.5 )</th>
<th>( x_i = 2.0 )</th>
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BIBLIOGRAPHY


